

Technical Appendix for “Frictional Wage Dispersion in Search Models: A Quantitative Assessment”

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This Technical Appendix to Hornstein, Krusell, and Violante (2007) (HKV, hereafter) contains a set of derivations that complement some of the results stated without proofs in our main paper. First, we show that introducing risk aversion into the standard search model will not affect results substantially, if workers can smooth consumption through access to a riskless asset (Section 5 in HKV). Second, we derive a detailed characterization of the search model with wage shocks during employment but no on-the-job search (Section 6 in HKV). Third, we develop a general model with on-the-job search, wage shocks and forced job-to-job mobility (Section 7.1 in HKV) and derive the Mm ratio. We use the version of the model to derive closed-form solutions for the average tenure and separation rate in the equilibrium of the standard model with on-the-job search (Section 7 in HKV). Fourth, we show how to compute bounds for the Mm ratio in the model with endogenous search effort on the job (Section 7.2). Finally, we describe how to derive the Mm ratio in simple versions of search models with counteroffers and wage tenure contracts (Section 7.3).

1 Search with risk aversion and access to credit markets

In Section 5 of HKV, we show that the basic search model with risk aversion but no income insurance is consistent with mean-min ratios of 1.7 only if workers are very risk averse: the degree of relative risk aversion must exceed eight. We also note that if workers have access to a risk-free bond for saving and borrowing then their behavior will be much closer to full insurance than to autarky. In the case where workers have constant absolute risk aversion (CARA) utility, one can defend this conclusion analytically.

Assume that workers have CARA preferences that satisfy , $u(c) = -\exp(-\theta c)$ with $\theta > 0$, and that they have access to a risk-free bond with interest rate r equal to their time discount factor. An unemployed worker receives income b , and at rate λ_u the worker receives wage offers from a distribution F . A worker that accepts a wage offer w will receive that wage until a fixed time period T .

Shimer and Werning (2006) derive the equation that characterizes the optimal reservation wage as

$$w^* = b + \frac{\lambda_u [1 - F(w^*)]}{r\sigma} \int_{w^*}^{\infty} \{1 + u(r\beta_T(w - w^*))\} \frac{dF(w)}{1 - F(w^*)}, \quad (1)$$

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where $\beta_T = \int_0^T e^{-rs} ds$. They also derive the optimal reservation wage when workers cannot borrow or save, and are forced to consume their income, delivering

$$u(w^*) = u(b) + \beta_T \lambda_u \int_{w^*}^{\infty} [u(w) - u(w^*)] dF(w). \quad (2)$$

We can rewrite these expressions for the reservation wage in terms of the replacement rate required to match a given Mm ratio. Normalizing the average wage, $\bar{w} \equiv 1$, equation (1) then becomes

$$\rho = \frac{1}{Mm} - \frac{\lambda_u^*}{r\sigma} \left[1 + \int_{w^*}^{\infty} u(r\beta_T(w - w^*)) \frac{dF(w)}{1 - F(w^*)} \right] \quad (3)$$

and equation (2) becomes

$$\rho = \frac{1}{Mm} - \frac{1}{\theta} \log \left\{ 1 + \beta_T \lambda_u^* \left[1 + \int_{w^*}^{\infty} u(w - w^*) \frac{dF(w)}{1 - F(w^*)} \right] \right\}. \quad (4)$$

In order to derive the implied replacement rates we need to evaluate the integral terms in equations (3) and (4). We consider two possibilities: a specification of the wage distribution as a Gamma function as in Section 4.6 of HKV, or a second-order approximation of the utility function as in Section 5 of HKV. For a Gamma function specification of the wage distribution, the integral term becomes

$$\int_{w^*}^{\infty} u(r\beta_T(w - w^*)) \frac{dF(w)}{1 - F(w^*)} = - \left[1 + r\beta_T \theta \alpha \left(1 - \frac{1}{Mm} \right) \right]^{-\gamma}. \quad (5)$$

Through a second-order approximation of the utility function $u(\cdot)$ around the mean of its argument $r\beta_T(\bar{w} - w^*)$, the integral term becomes

$$\int_{w^*}^{\infty} u(r\beta_T(w - w^*)) \frac{dF(w)}{1 - F(w^*)} = - \exp \left[-r\beta_T \theta \left(1 - \frac{1}{Mm} \right) \right] \left[1 + \frac{1}{2} (r\beta_T \theta cv)^2 \right] \quad (6)$$

where cv is the coefficient of variation.

For a quantitative evaluation of the model, we now derive the replacement ratios that are needed to match the mean-min ratio $Mm = 1.7$, for various degrees of risk aversion. This exercise is conditional on the interest rate $r = 0.0042$, the effective job finding rate $\lambda_u^* = 0.39$, the unemployment rate $u = 0.0488$, and a coefficient of variation $cv = 0.3$ for wages. With a fixed job duration T , the steady state unemployment rate is

$$u = 1 / (1 + \lambda_u^* T),$$

and the implied job duration is

$$T = \frac{1 - u}{u} \frac{1}{\lambda_u^*} = 49$$

months, i.e., somewhat over four years.

In Figure 1 we graph the required replacement ratio with a second-order approximation for the utility function in the integral term, and the required replacement ratio with a Gamma-function specification of the wage distribution. Our results are independent of the approximation we choose.

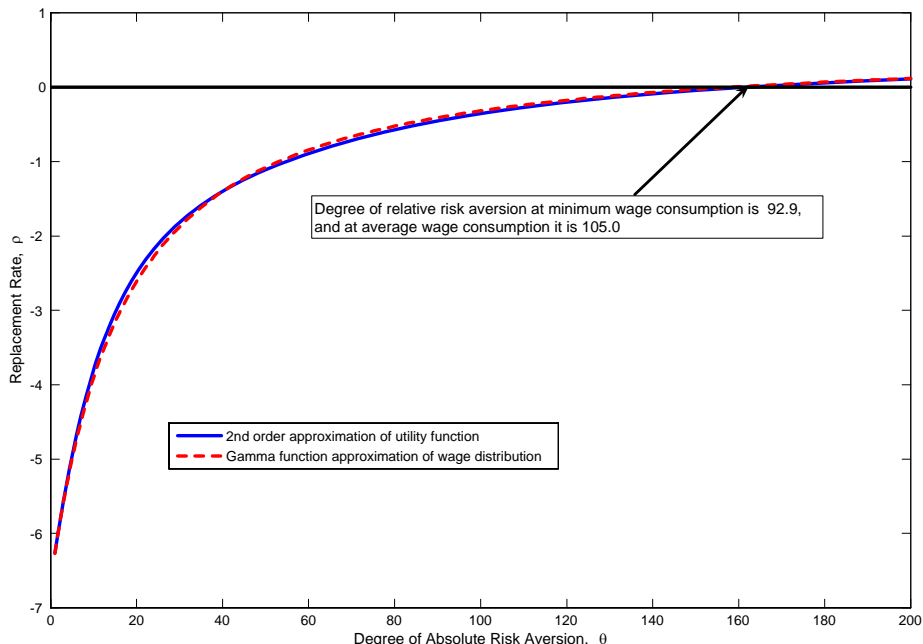


Figure 1. Replacement rates ρ to match the Mm ratio when workers can borrow and save

When workers can borrow and save at a constant interest rate, then we cannot match the observed Mm ratio with positive replacement ratio, unless the degree of absolute risk aversion exceeds 150, implying a degree of relative risk aversion of 105 for a worker with average wages and zero net asset holding.

2 Search with wage shocks during employment

In this section, we characterize the equilibrium of the search model with wage shocks while employed when wage shocks when employed and unemployed come from the same distribution. We first state the discrete-time approximation of the search model for a fixed and finite length of the time period. We then derive the continuous-time representation as the limit of the discrete-time approximation when the length of the time period becomes arbitrarily small. We show that the Bellman equations for the employment and unemployment values for the continuous- and discrete-time version are the same. Our derivation of the mean-min ratio for wage inequality is therefore independent of the time representation. We then show that in the discrete-time version the first-order autocorrelation coefficient of wages is one minus the arrival rate of wage changes. Finally, we consider a variation of the baseline model where wage shocks when employed come from a different distribution than wage shocks when unemployed. In particular, we study the Mortensen-Pissarides (1994) environment where unemployed workers on meeting a job always receive the highest wage.

We describe the parameters of the environment in terms of the continuous-time framework. The economy is populated by ex-ante equal, risk-neutral, infinitely lived individuals who discount the future at rate r . Unemployed agents receive job offers at the instantaneous rate λ_u , and the wage of employed agents changes at the instantaneous rate δ . Conditionally on receiving a wage change, the wage is drawn from a well-behaved distribution function

$F(w)$ with upper support w^{\max} . Draws are i.i.d. over time and across agents. Note that employed and unemployed agents sample from the same wage distribution. If a job offer w is accepted, the worker is paid a wage w , until the next wage-change event occurs. While unemployed, the worker receives a utility flow b which includes unemployment benefits and a value of leisure and home production, net of search costs. We only consider steady-state allocations.

2.1 Discrete time versus continuous time

The discrete-time approximations of the Bellman equations for the value of employment, $W(w)$, and unemployment, U , are

$$W(w) = w\Delta + e^{-r\Delta} \left\{ \delta\Delta \int \max\{W(z), U\} dF(z) + (1 - \delta\Delta) W(w) \right\}, \quad (7)$$

$$U = b\Delta + e^{-r\Delta} \left\{ \lambda_u\Delta \int \max\{W(z), U\} dF(z) + (1 - \lambda_u\Delta) U \right\}, \quad (8)$$

where Δ is the length of the time interval, and $\lambda_u\Delta$ ($\delta\Delta$) is the probability that an (un)employed worker receives a wage offer at the end of the interval Δ . Using the definition of the reservation wage, $W(w^*) = U$, and rearranging terms the value equations can be rewritten as

$$(1 - e^{-r\Delta}) W(w) = w\Delta + e^{-r\Delta} \left\{ \delta\Delta \int_{w^*}^{w^{\max}} [W(z) - W(w)] dF(z) - \delta\Delta F(w^*) [W(w) - U] \right\},$$

$$(1 - e^{-r\Delta}) U = b\Delta + e^{-r\Delta} \left\{ \lambda_u\Delta \int_{w^*}^{w^{\max}} [W(z) - U] dF(z) \right\}$$

Dividing by the length of the time interval and taking the limit as $\Delta \rightarrow 0$ we obtain the continuous-time Bellman equations

$$rW(w) = w + \delta \int_{w^*}^{w^{\max}} [W(z) - W(w)] dF(z) - \delta F(w^*) [W(w) - U], \quad (9)$$

$$rU = b + \lambda_u \int_{w^*}^{w^{\max}} [W(z) - U] dF(z). \quad (10)$$

Rather than studying the continuous-time limit of the search model with wage changes, we can just use the discrete-time approximation and consider a unit length interval, $\Delta = 1$. In this case we obtain the following expressions for the value functions of being employed and unemployed:

$$(1 - \beta) W(w) = w + \beta \left\{ \delta \int_{w^*}^{w^{\max}} [W(z) - W(w)] dF(z) - \delta F(w^*) [W(w) - U] \right\}$$

$$(1 - \beta) U = b + \beta \lambda_u \int_{w^*}^{w^{\max}} [W(z) - U] dF(z),$$

where $\beta \equiv e^{-r} \equiv 1/(1 + \bar{r})$. Note that we can rewrite these discrete-time value equations as

$$\bar{r}\bar{W}(w) = w + \delta \int_{w^*}^{w^{\max}} [\bar{W}(z) - \bar{W}(w)] dF(z) - \delta F(w^*) [\bar{W}(w) - \bar{U}] \quad (11)$$

$$\bar{r}\bar{U} = b + \lambda_u \int_{w^*}^{w^{\max}} [\bar{W}(z) - \bar{U}] dF(z), \quad (12)$$

where $\bar{W} \equiv W/(1 + \bar{r})$ and $\bar{U} \equiv U/(1 + \bar{r})$. Expressions (11) and (12) are formally equivalent to the expressions (9) and (10) for the continuous-time value functions. Therefore, the results for the mean-min ratio also apply for the discrete-time version of the paper.

2.2 The reservation wage

As a first step towards deriving the mean-min wage ratio for the continuous-time model we characterize the reservation wage. For this purpose, evaluate the employment value expression (9) at w^* and use the definition of the reservation wage, $W(w^*) = U$, to deliver

$$rU = w^* + \delta \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z).$$

Now substitute the unemployment value expression (10) for the left-hand side and solve for the reservation wage

$$w^* = b + (\lambda_u - \delta) \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z). \quad (13)$$

Integration by parts on the right-hand side yields

$$\begin{aligned} \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z) &= [(W(z) - W(w^*)) F(z)]_{w^*}^{w^{\max}} - \int_{w^*}^{w^{\max}} W'(z) F(z) dz \\ &= W(w^{\max}) - W(w^*) - \int_{w^*}^{w^{\max}} W'(z) F(z) dz \\ &= \int_{w^*}^{w^{\max}} W'(z) [1 - F(z)] dz. \end{aligned} \quad (14)$$

From the employment value expression (9) it follows that

$$W'(w) = \frac{1}{r + \delta}. \quad (15)$$

Hence the reservation-wage expression is

$$\begin{aligned} w^* &= b + \frac{\lambda_u - \delta}{r + \delta} \int_{w^*}^{w^{\max}} [1 - F(z)] dz \\ &= b + \frac{(\lambda_u - \delta) [1 - F^*]}{r + \delta} \int_{w^*}^{w^{\max}} \left[\frac{1}{1 - F^*} - \frac{F(z)}{1 - F^*} \right] dz \end{aligned} \quad (16)$$

with $F^* = F(w^*)$.

2.3 The equilibrium wage distribution

We now construct the equilibrium wage distribution $G(w)$ implied by the interaction of the wage-offer distribution and the reservation wage. The measure of agents with wage below w is $(1 - u)G(w)$. Agents leave this stock because their wage changes and their new wage is either less than the reservation wage or higher than the current wage. Agents enter this stock if they were unemployed and receive an acceptable wage offer below w , or if they were employed at wage above w and are forced to accept a lower wage; hence

$$\begin{aligned} &(1 - u)G(w)\delta\{F(w^*) + [1 - F(w)]\} \\ &= \{u\lambda_u + (1 - u)[1 - G(w)]\delta\}[F(w) - F(w^*)]. \end{aligned} \quad (17)$$

We can solve this expression for the equilibrium wage distribution as a function of the wage-offer distribution:

$$G(w) = \left[\frac{\lambda_u u}{\delta(1 - u)} + 1 \right] [F(w) - F(w^*)]. \quad (18)$$

In steady state, the inflows and outflows from employment balance:

$$(1 - u) \delta F(w^*) = u \lambda_u [1 - F(w^*)].$$

Using the expression for steady-state employment in (18) we obtain

$$G(w) = \frac{F(w) - F(w^*)}{1 - F(w^*)}. \quad (19)$$

Thus, the equilibrium wage distribution with and without wage shocks during employment are the same, namely the wage-offer distribution truncated at the reservation wage.

2.4 The mean-min ratio

Based on the equilibrium wage distribution we can calculate the average wage of employed workers as

$$\bar{w} = \int_{w^*}^{w^{\max}} w \frac{dF(w)}{1 - F^*} = \frac{w^{\max} - w^* F^*}{1 - F^*} - \int_{w^*}^{w^{\max}} \frac{F(z)}{1 - F^*} dz. \quad (20)$$

Solving the average-wage expression (20) for the right-hand side integral term and substituting this term for the corresponding integral in the reservation-wage expression (16) yields

$$\begin{aligned} w^* &= b + \frac{(\lambda_u - \delta)[1 - F^*]}{r + \delta} \left[\int_{w^*}^{w^{\max}} \frac{1}{1 - F^*} dz + \bar{w} - \frac{w^{\max} - w^* F^*}{1 - F^*} \right] \\ &= b + \frac{(\lambda_u - \delta)[1 - F^*]}{r + \delta} [\bar{w} - w^*]. \end{aligned} \quad (21)$$

Using the definition of the replacement rate, $b = \rho \bar{w}$, we can solve equation (21) for the reservation wage and obtain an expression for the mean-min ratio, that is, the ratio of average wages to the reservation wage,

$$\frac{\bar{w}}{w^*} = \frac{\frac{(\lambda_u - \delta)[1 - F(w^*)]}{r + \delta} + 1}{\frac{(\lambda_u - \delta)[1 - F(w^*)]}{r + \delta} + \rho} = \frac{\frac{\lambda_u^* - \delta + \sigma^*}{r + \delta} + 1}{\frac{\lambda_u^* - \delta + \sigma^*}{r + \delta} + \rho}, \quad (22)$$

with $\lambda_u^* \equiv (1 - F^*) \lambda_u$ and $\sigma^* \equiv \delta F^*$. This is equation (16) in Section 6 of HKV. Note that as δ goes to infinity $\frac{\bar{w}}{w^*}$ goes to $1/\rho$.

2.5 Wage persistence in the discrete-time model

It is straightforward to show that the equilibrium wage distribution for the discrete-time and the continuous-time versions of the model are the same; we now work with the former. We need the expected value of the cross-product of today's and tomorrow's wage, conditional on being employed in both periods, to calculate the autocorrelation coefficient. We proceed in two steps: first, we obtain the value conditional on today's wage, and then we integrate over today's wage to obtain the unconditional expectation. This delivers

$$\begin{aligned} E[w'w|w] &= (1 - \delta) w^2 + \delta w E[\tilde{w} | \tilde{w} \geq w^*] \\ &= (1 - \delta) w^2 + \delta w \bar{w} \\ E[w'w] &= (1 - \delta) E[w^2] + \delta \bar{w}^2. \end{aligned}$$

We can now define the first-order autocorrelation coefficient as

$$\begin{aligned}\rho &= \frac{(1 - \delta) E[w^2] + \delta \bar{w}^2 - \bar{w}^2}{Var(w)} \\ &= \frac{(1 - \delta) (E[w^2] - \bar{w}^2)}{Var(w)} \\ &= 1 - \delta.\end{aligned}$$

2.6 The mean-min ratio for a Mortensen-Pissarides (1994) environment

Suppose now that an unemployed worker who receives a wage offer always receives the highest wage w^{\max} , whereas wage changes of employed workers continue to be drawn from the distribution F . We will show that the mean-min ratio that we previously derived for our baseline model, equation (22), represents an upper bound for the mean-min ratio in the Mortensen-Pissarides (1994) environment.

The value function equation for an employed worker, (9), remains unchanged, but the value function equation for an unemployed worker is now

$$rU = b + \lambda_u [W(w^{\max}) - U]. \quad (23)$$

Following the same steps as in Section 2.2 we derive the modified expression for the reservation wage:

$$w^* = b + \frac{\lambda_u - \delta}{r + \delta} [w^{\max} - w^*] + \frac{\delta}{r + \delta} \int_{w^*}^{w^{\max}} F(z) dz. \quad (24)$$

The modified steady-state expression characterizing the equilibrium wage distribution is now

$$\begin{aligned}(1 - u) G(w) \delta \{F(w^*) + [1 - F(w)]\} \\ = (1 - u) [1 - G(w)] \delta [F(w) - F(w^*)] \text{ for } w < w^{\max}.\end{aligned} \quad (25)$$

Note that there are no inflows from the pool of unemployed since all unemployed workers who receive wage offers receive the highest wage. Thus, the equilibrium wage distribution for $w < w^{\max}$ is

$$G(w) = F(w) - F(w^*). \quad (26)$$

Since all unemployed workers receive the highest wage with probability one there is now a mass point at $w = w^{\max}$, that is, the cumulative density function is discontinuous at w^{\max} .

Integrating the wage with respect to the equilibrium wage distribution then yields the average wage

$$\bar{w} = \int_{w^*}^{w^{\max}} w dF(w) + w^{\max} F(w^*) = w^{\max} + F(w^*) (w^{\max} - w^*) - \int_{w^*}^{w^{\max}} F(w) dw. \quad (27)$$

Solving the average-wage expression (27) for the right-hand side integral term and substituting this term for the corresponding integral in the reservation-wage expression (24) yields

$$\begin{aligned}\left[1 + \frac{\lambda_u - (1 - F^*) \delta}{r + \delta}\right] w^* &= \left(\rho - \frac{\delta}{r + \delta}\right) \bar{w} + \frac{\lambda_u + \delta F^*}{r + \delta} w^{\max} \\ &> \left(\rho + \frac{\lambda_u - \delta (1 - F^*)}{r + \delta}\right) \bar{w}.\end{aligned} \quad (28)$$

Note that the last inequality implies that the mean-min ratio for this setup is bounded above by that of the baseline economy with wage shocks given in (22).

3 On-the-job search with reallocation and wage shocks

We now describe a general version of the on-the-job search model that includes forced job-to-job mobility (reallocation shocks) and wage shocks (the models in Section 7.1 of HKV). The Bellman equations for the employment and unemployment values are

$$\begin{aligned} rW(w) &= w + \lambda_e \int \max\{W(z) - W(w), 0\} dF(z) - \sigma [W(w) - U] \\ &\quad + \phi \int [\max\{W(z), U\} - W(w)] dF(z) \\ rU &= b + \lambda_u \int \max\{W(z) - U, 0\} dF(z). \end{aligned}$$

The basic on-the-job search model in HKV, Section 7, assumes that a worker receives outside wage offers at a rate λ_e . Without loss of generality, and motivated by what equilibrium firm behavior would dictate, we assume that the wage-offer distribution is such that unemployed workers accept all wage offers: $F(w^*) = 0$. A worker can always reject a wage offer and keep the current wage. The worker may lose the current job at an exogenous separation rate σ .

On-the-job search with forced job-to-job mobility in HKV, Section 7.1, assumes that for some wage offers, the worker just has to take the offer, even if the new job pays less than the current job. Forced job mobility is reflected in the parameter $\phi > 0$. In the basic on-the-job search model $\phi = 0$.

On-the-job search with wage shocks (also discussed in HKV, Section 7.1), is isomorphic to forced job-to-job mobility as far as the wage equilibrium distribution is concerned. The only difference is in its implications for observable transitions. The arrival of a type ϕ wage offer now results in a wage cut on the existing job rather than in a separation and transfer to a lower-paying job.

3.1 The reservation wage

Workers continue to follow reservation-wage strategies and the Bellman equations can be rewritten as

$$\begin{aligned} rW(w) &= w + \lambda_e \int_w^{w^{\max}} [W(z) - W(w)] dF(z) - \sigma [W(w) - U] \\ &\quad + \phi \int_{w^*}^{w^{\max}} [W(z) - W(w)] dF(z) \\ &= w + (\lambda_e + \phi) \int_w^{w^{\max}} [W(z) - W(w)] dF(z) - \sigma [W(w) - U] \\ &\quad + \phi \int_{w^*}^w [W(w) - W(z)] dF(z) \end{aligned} \tag{29}$$

$$rU = b + \lambda_u \int_{w^*}^{w^{\max}} [W(z) - U] dF(z). \tag{30}$$

Evaluate the employment value equation (29) at w^* , using the reservation-wage property, $W(w^*) = U$, and the unemployment value expression (30) to obtain

$$\begin{aligned} rU &= w^* + (\lambda_e + \phi) \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z) \\ &= b + \lambda_u \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z). \end{aligned}$$

We can solve this expression for the reservation wage

$$w^* = b + (\lambda_u - \lambda_e - \phi) \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z). \quad (31)$$

As with equation (13), we can integrate the right-hand side integral by parts, as in (14), and obtain the reservation-wage equation

$$w^* = b + (\lambda_u - \lambda_e - \phi) \int_{w^*}^{w^{\max}} W'(z) [1 - F(z)] dz. \quad (32)$$

Note that differentiating the employment value equation (29) with respect to the current wage yields

$$W'(w) = \frac{1}{r + \sigma + \phi + \lambda_e [1 - F(w)]}. \quad (33)$$

Substituting (33) in (32) we can rewrite the reservation wage as

$$w^* = b + (\lambda_u - \lambda_e - \phi) \int_{w^*}^{w^{\max}} \frac{1 - F(z)}{r + \sigma + \phi + \lambda_e [1 - F(z)]} dz. \quad (34)$$

3.2 The equilibrium wage distribution

We now construct the equilibrium wage distribution $G(w)$ implied by the interaction of the wage-offer distribution and the reservation wage.

The measure of agents with wage below w is $(1 - u)G(w)$. Agents leave this stock because (1) they are separated at rate σ , (2) they receive an outside offer which they accept at rate $\lambda_e [1 - F(w)]$, or (3) they are forced to leave at rate ϕ but are lucky enough to obtain an offer above w . Workers enter this stock if (1) they were unemployed and receive a wage offer below w or (2) they were employed at wage above w and are forced to accept a lower wage. In a steady state the inflows and outflows balance:

$$\begin{aligned} & (1 - u)G(w) \{ \sigma + (\lambda_e + \phi) [1 - F(w)] \} \\ & = u\lambda_u F(w) + \phi(1 - u) [1 - G(w)] F(w). \end{aligned} \quad (35)$$

We can solve this expression for the equilibrium wage distribution as a function of the wage-offer distribution:

$$G(w) = \frac{\lambda_u u + \phi(1 - u)}{1 - u} \cdot \frac{F(w)}{\sigma + \phi + \lambda_e [1 - F(w)]}. \quad (36)$$

In steady state, if all the job offers are above w^* so that $F(w^*) = 0$,

$$u\lambda_u = (1 - u)\sigma.$$

Hence

$$G(w) = \frac{(\sigma + \phi) F(w)}{\sigma + \phi + \lambda_e [1 - F(w)]} \quad (37)$$

and

$$\begin{aligned} 1 - G(w) &= \frac{\sigma + \phi + \lambda_e}{\sigma + \phi + \lambda_e [1 - F(w)]} [1 - F(w)] \\ &\simeq \frac{r + \sigma + \phi + \lambda_e}{r + \sigma + \phi + \lambda_e [1 - F(w)]} [1 - F(w)]. \end{aligned} \quad (38)$$

3.3 The mean-min ratio

The average wage is

$$\begin{aligned}
\bar{w} &= \int_{w^*}^{w^{\max}} w dG(z) = [wG(w)]_{w^*}^{w^{\max}} - \int_{w^*}^{w^{\max}} G(z) dz \\
&= w^{\max} - \int_{w^*}^{w^{\max}} G(z) dz \\
&= [w^{\max} - w^*] + w^* - \int_{w^*}^{w^{\max}} G(z) dz \\
&= w^* + \int_{w^*}^{w^{\max}} [1 - G(z)] dz.
\end{aligned} \tag{39}$$

Solve the wage distribution expression (38) for $1 - F$ and use it in the reservation-wage expression (34) to obtain

$$w^* \simeq b + \frac{\lambda_u - \lambda_e - \phi}{r + \sigma + \phi + \lambda_e} \int_{w^*}^{w^{\max}} [1 - G(z)] dz.$$

Finally substituting for the integral term from the average-wage equation (39) we can solve for the mean-min ratio:

$$\begin{aligned}
w^* &\simeq \rho \bar{w} + \frac{\lambda_u - \lambda_e - \phi}{r + \sigma + \phi + \lambda_e} (\bar{w} - w^*) \quad \Rightarrow \\
Mm &\simeq \frac{\frac{\lambda_u - \lambda_e - \phi}{r + \sigma + \phi + \lambda_e} + 1}{\frac{\lambda_u - \lambda_e - \phi}{r + \sigma + \phi + \lambda_e} + \rho}.
\end{aligned} \tag{40}$$

Equation (40) corresponds to equations (15) and (18) in Section 7.1 of HKV.

We now show how to derive closed-form solutions for the average tenure and separation rate in the equilibrium of the standard model with on-the-job search (Section 7 in HKV).

3.4 Turnover rates in the basic on-the-job search model

We show how to derive closed-form solutions for the average tenure and separation rate in the equilibrium of the standard model with on the job search. Consider, for simplicity, the version of the on-the-job search model without reallocation shocks ($\phi = 0$). This is the model studied in Section 7 of HKV. Incorporating the reallocation shocks in our derivations below is immediate.

The average *completed job tenure* in the baseline model with on-the-job search is

$$\tau = \int_{w^*}^{w^{\max}} \frac{dG(w)}{\sigma + \lambda_e [1 - F(w)]}. \tag{41}$$

Recall the equilibrium wage distribution from (37) and solve that expression for the wage-offer distribution

$$1 - F(w) = \frac{\sigma [1 - G(w)]}{\sigma + \lambda_e G(w)}.$$

Substitute this expression in the expression for average job tenure (41) and we arrive at

$$\begin{aligned}
\tau &= \int_{w^*}^{w^{\max}} \frac{dG(w)}{\sigma + \lambda_e \frac{\sigma [1 - G(w)]}{\sigma + \lambda_e G(w)}} = \int_{w^*}^{w^{\max}} \frac{[\sigma + \lambda_e G(w)] dG(w)}{\sigma (\lambda_e + \sigma)} \\
&= \frac{1}{\sigma (\lambda_e + \sigma)} \left[\sigma + \lambda_e \int_{w^*}^{w^{\max}} G(w) dG(w) \right].
\end{aligned} \tag{42}$$

Note that

$$\int_{w^*}^{w^{\max}} G(w) dG(w) = [G^2(w)]_{w^*}^{w^{\max}} - \int_{w^*}^{w^{\max}} G(w) dG(w).$$

Thus

$$\int_{w^*}^{w^{\max}} G(w) dG(w) = 1/2.$$

Hence, the average job tenure is

$$\tau = \frac{\sigma + \lambda_e/2}{\sigma(\lambda_e + \sigma)} \in \left(\frac{1}{2\sigma}, \frac{1}{\sigma} \right). \quad (43)$$

The (total) *separation rate* for employed workers, the sum of exogenous separations and job-to-job transitions, is

$$\chi = \sigma + \lambda_e \int_{w^*}^{w^{\max}} [1 - F(w)] dG(w). \quad (44)$$

Hence, the share of separations attributable to job-to-job separations is

$$\frac{jjssep}{\chi} = \frac{\lambda_e \int_{w^*}^{w^{\max}} [1 - F(w)] dG(w)}{\sigma + \lambda_e \int_{w^*}^{w^{\max}} [1 - F(w)] dG(w)}. \quad (45)$$

Following Nagypal (2005), and integrating *jjssep* by parts, yields

$$\begin{aligned} jjssep &= \lambda_e \int_{w^*}^{w^{\max}} [1 - F(w)] dG(w) \\ &= \lambda_e - \lambda_e \int_{w^*}^{w^{\max}} F(w) dG(w) \\ &= \lambda_e - \lambda_e [F(w) G(w)]_{w^*}^{w^{\max}} + \lambda_e \int_{w^*}^{w^{\max}} G(w) dF(w) \\ &= \lambda_e \int_{w^*}^{w^{\max}} G(w) dF(w) \\ &= \lambda_e \sigma \int_{w^*}^{w^{\max}} \frac{F(w)}{\sigma + \lambda_e [1 - F(w)]} dF(w). \end{aligned} \quad (46)$$

The last step uses equation (37) for G . Now change the variable of integration to $z = F(w)$, and we obtain

$$\begin{aligned} \int_0^1 \frac{z}{\sigma + \lambda_e [1 - z]} dz &= -\frac{\lambda_e \{z\}_0^1 + (\lambda_e + \sigma) \{\log[\sigma + \lambda_e(1 - z)]\}_0^1}{\lambda_e^2} \\ &= -\frac{\lambda_e + (\lambda_e + \sigma) (\log \sigma - \log(\sigma + \lambda_e))}{\lambda_e^2} \\ &= \frac{(\lambda_e + \sigma) \log\left(\frac{\sigma + \lambda_e}{\sigma}\right)}{\lambda_e^2} - \frac{1}{\lambda_e}. \end{aligned} \quad (47)$$

Hence, the job-to-job separation rate is

$$jjssep = \frac{\sigma(\lambda_e + \sigma) \log\left(\frac{\sigma + \lambda_e}{\sigma}\right)}{\lambda_e} - \sigma \quad (48)$$

and the total separation rate is

$$\chi = jjssep + \sigma = \frac{\sigma(\lambda_e + \sigma) \log\left(\frac{\sigma + \lambda_e}{\sigma}\right)}{\lambda_e}. \quad (49)$$

3.5 Wage cuts in the model with wage shocks

In this version of the model, not only does a worker receive outside wage offers at the rate λ_e , but the wage on the current job is also changing with the arrival rate ϕ . If the new wage is less than the current wage, the worker has to accept a pay cut. Relative to the model with reallocation shocks, this event leads to the same change in the wage but to no job-to-job transition.

The average rate at which workers are forced to take a wage cut is

$$\begin{aligned}\zeta &= \phi \int_{w^*}^{w^{\max}} F(w) dG(w) \\ &= \phi [F(w) G(w)]_{w^*}^{w^{\max}} - \phi \int_{w^*}^{w^{\max}} G(w) dF(w) \\ &= \phi - \phi \int_{w^*}^{w^{\max}} G(w) dF(w).\end{aligned}\tag{50}$$

Now substitute for the equilibrium wage distribution from (37) and we obtain

$$\zeta = \phi - \phi \hat{\sigma} \int_{w^*}^{w^{\max}} \frac{F(w)}{\hat{\sigma} + \lambda_e [1 - F(w)]} dF(w),\tag{51}$$

where $\hat{\sigma} \equiv \sigma + \phi$. We have previously derived the right-hand side integral: expressions (46) and (47). Use equation (47) for the integral and we arrive at

$$\begin{aligned}\zeta &= \phi - \phi \hat{\sigma} \left[\frac{(\lambda_e + \hat{\sigma}) \log\left(\frac{\hat{\sigma} + \lambda_e}{\hat{\sigma}}\right)}{\lambda_e^2} - \frac{1}{\lambda_e} \right] \\ &= \phi \left[1 + \frac{\hat{\sigma}}{\lambda_e} - \frac{\hat{\sigma} (\lambda_e + \hat{\sigma}) \log\left(\frac{\hat{\sigma} + \lambda_e}{\hat{\sigma}}\right)}{\lambda_e^2} \right].\end{aligned}\tag{52}$$

With ζ being the average arrival rate of a wage cut, the fraction of employed workers that receive a wage cut in a unit time period is $1 - \exp(-\zeta)$.

4 On-the-job search with endogenous effort choice

We now derive the results stated in Section 7.2 formally. Based on the derivations in Mortensen (2007), we generalize the Christensen et al. (2005) model in order to allow for asymmetric search cost functions off and on the job. Let the utility cost of attaining a contact rate λ_i in employment state $i \in \{u, e\}$ be $c_i(\lambda_i)$, with $c'_i > 0$ and $c''_i > 0$. In particular, we will assume that the search cost function is isoelastic: $c_i(\lambda_i) = \kappa_i \lambda_i^{1+1/\gamma}$, with $\gamma > 0$.¹ Optimal search effort choice will deliver a scalar λ_u^o for the unemployed worker and a policy function $\lambda_e^o(w)$ for the employed worker.

A worker employed at a job with wage w accepts any job offer with a higher wage. The flow values of unemployment and employment are, respectively,

$$rU = b - c_u(\lambda_u^o) + \lambda_u^o \int_{w^*}^{w^{\max}} [W(z) - U] dF(z),\tag{53}$$

$$rW(w) = w - c_w[\lambda_e^o(w)] + \lambda_e^o(w) \int_w^{w^{\max}} [W(z) - W(w)] dF(z) - \sigma [W(w) - U].\tag{54}$$

¹The parameter γ is not to be confused with the elasticity parameter in the Gamma function of Section 1.

The FOC for optimal search effort for an unemployed worker and for a worker employed at wage w that characterize the search effort choices $\{\lambda_u^o, \lambda_e^o(w)\}$ are, respectively,

$$c'_e[\lambda_e^o(w)] = \int_w^{w^{\max}} [W(z) - W(w)] dF(z) \quad (55)$$

$$c'_u(\lambda_u^o) = \int_{w^*}^{w^{\max}} [W(z) - U] dF(z). \quad (56)$$

By simple inspection of (55), it is immediate that $\lambda_e^o(w)$ is decreasing in w .

In steady state, flows for the workers employed at wage w or lower satisfy

$$(1-u)G(w)\sigma + (1-u)[1-F(w)] \int_{w^*}^w \lambda_e^o(z) dG(z) = u\lambda_u^o F(w), \quad (57)$$

and flows in and out of unemployment satisfy the relation

$$u\lambda_u^o = (1-u)\sigma.$$

Combining the last two equations we obtain

$$\sigma F(w) = G(w)\sigma + [1-F(w)] G(w) \bar{\lambda}_e(w),$$

where

$$\bar{\lambda}_e(w) \equiv \frac{\int_{w^*}^w \lambda_e^o(w) dG(w)}{G(w)}$$

is the average search effort for workers with wages less than w . From (57) it is easy to see that the equilibrium wage distribution $G(w)$ can be expressed as a function of the wage-offer distribution $F(w)$, i.e., through

$$1 - G(w) = \frac{[\sigma + \bar{\lambda}_e(w)] [1 - F(w)]}{\sigma + \bar{\lambda}_e(w) [1 - F(w)]} \simeq \frac{[r + \sigma + \bar{\lambda}_e(w)] [1 - F(w)]}{r + \sigma + \bar{\lambda}_e(w) [1 - F(w)]} \quad (58)$$

where the approximation sign comes from $r \simeq 0$. This is exactly the same approximation as the one we used in deriving equation (38).

Note that the right-hand side of (58) is increasing in $\bar{\lambda}_e(w)$. Since we have derived that search effort is decreasing in w , we can establish the following ranking:

$$\lambda_{w^*} \geq \bar{\lambda}_e(w) \geq \lambda_w \geq \lambda_{w^{\max}} = 0,$$

where, for notational simplicity, we have denoted $\lambda_e^o(w)$ by λ_w . The above inequalities, in conjunction with (58), imply

$$\begin{aligned} \frac{(r + \sigma + \lambda_{w^*}) [1 - F(w)]}{r + \sigma + \lambda_w [1 - F(w)]} &\geq 1 - G(w) \geq \frac{(r + \sigma + \lambda_w) [1 - F(w)]}{r + \sigma + \lambda_w [1 - F(w)]} \\ &\geq \frac{(r + \sigma + \lambda_{w^{\max}}) [1 - F(w)]}{r + \sigma + \lambda_w [1 - F(w)]}. \end{aligned} \quad (59)$$

Now we derive the reservation-wage equation for this model. Going back to the value function in (54), the derivative of the worker value function (using the Envelope Theorem) is

$$W'(w) = \frac{1}{r + \sigma + \lambda_w [1 - F(w)]}.$$

Integrating by parts and using the derivative above yields

$$\int_w^{w^{\max}} [W(z) - W(w)] dF(z) = \int_w^{w^{\max}} \frac{[1 - F(z)]}{r + \sigma + \lambda_z [1 - F(z)]} dz. \quad (60)$$

The definition of the reservation wage, $W(w^*) = U$, implies that

$$\begin{aligned} w^* - c_e(\lambda_{w^*}) + \lambda_{w^*} \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z) \\ = b - c_u(\lambda_u) + \lambda_u \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z), \end{aligned} \quad (61)$$

where for notational simplicity we denoted λ_u^o by λ_u . Since the search cost functions are isoelastic, from (55) we obtain

$$c_e(\lambda_{w^*}) = \kappa_e \lambda_{w^*}^{1+1/\gamma} = \lambda_{w^*} \left(\frac{\gamma}{1+\gamma} \right) c'_e(\lambda_{w^*}) = \lambda_{w^*} \left(\frac{\gamma}{1+\gamma} \right) \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z) \quad (62)$$

for the optimal search cost when employed; similarly, from (56) we arrive at

$$c_u(\lambda_u) = \lambda_u \left(\frac{\gamma}{1+\gamma} \right) \int_{w^*}^{w^{\max}} [W(z) - W(w^*)] dF(z) \quad (63)$$

for the optimal search cost when unemployed. Substituting (63) and (62) into (61) and using (60) we arrive at

$$w^* = b + \frac{\lambda_u - \lambda_{w^*}}{1+\gamma} \int_{w^*}^{w^{\max}} \frac{1 - F(z)}{r + \sigma + \lambda_z [1 - F(z)]} dz. \quad (64)$$

Now we use the inequalities in (59) to construct bounds for the Mm ratio. Putting together (64) and (59) we obtain

$$\begin{aligned} w^* &\geq \rho \bar{w} + \frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma + \lambda_{w^*})} \int_{w^*}^{w^{\max}} [1 - G(z)] dz \\ &= \rho \bar{w} + \frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma + \lambda_{w^*})} [\bar{w} - w^*] \end{aligned} \quad (65)$$

and, similarly,

$$\begin{aligned} w^* &\leq \rho \bar{w} + \frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma + \lambda_{w^{\max}})} \int_{w^*}^{w^{\max}} [1 - G(z)] dz \\ &= \rho \bar{w} + \frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma)} [\bar{w} - w^*]. \end{aligned} \quad (66)$$

Inequalities (65) and (66) yield the following bounds for the Mm ratio

$$\frac{\frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma)} + 1}{\frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma)} + \rho} \leq Mm \leq \frac{\frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma + \lambda_{w^*})} + 1}{\frac{\lambda_u - \lambda_{w^*}}{(1+\gamma)(r + \sigma + \lambda_{w^*})} + \rho}.$$

We can proceed in a similar manner to construct bounds on search costs. Combining (63) with (60), and using the inequalities in (59), we arrive at

$$\frac{\lambda_u}{r + \sigma} \left(\frac{\gamma}{1+\gamma} \right) \left(1 - \frac{1}{Mm} \right) \geq \frac{c_u(\lambda_u)}{\bar{w}} \geq \frac{\lambda_u}{r + \sigma + \lambda_{w^*}} \left(\frac{\gamma}{1+\gamma} \right) \left(1 - \frac{1}{Mm} \right),$$

which is the expression reported in equation (25) of the main text.

5 On-the-job search with counteroffers

This model, discussed in Section 7.3, is a simplified version of that presented in Mortensen (2005). Consider an economy with only one type of firm with productivity p . If the worker is contacted when unemployed, the hiring firm pays a wage w^* that makes the worker exactly indifferent between searching further and working at w^* . However, if an employed worker is contacted by another firm, the two firms compete in a Bertrand fashion and bid up her wage to p .

The flow value of unemployment is

$$rU = b + \lambda_u (W_0 - U), \quad (67)$$

where W_0 is the value of a newly-employed worker, with

$$rW_0 = w^* + \lambda_e (W_1 - W_0) - \sigma (W_0 - U). \quad (68)$$

The value W_1 is the value of employment after being contacted by another firm. Because of Bertrand competition among firms, after the contact, the wage jumps to p , and hence

$$rW_1 = p - \sigma (W_1 - U). \quad (69)$$

The reservation wage w^* satisfies $W_0 = U$. Combining (67), (68), and (69) we obtain the reservation-wage equation

$$w^* = b - \frac{\lambda_e}{r + \sigma} (p - b)$$

in Section 7.3 of the main text.

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