

Sovereign Default and Debt Renegotiation

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Abstract

We develop a small open economy model to study sovereign default and debt renegotiation. The model features both endogenous default risk and endogenous debt recovery rates. Sovereign bonds are priced to compensate creditors for the risk of default and the risk of debt restructuring. We find that both equilibrium debt recovery rates and sovereign bond prices decrease with the level of debt. In a quantitative analysis, the model successfully accounts for the volatile and countercyclical bond spreads, countercyclical current account and other empirical regularities of the Argentine economy. The model also replicates the dynamics of bond spreads during the recent debt crisis in Argentina.

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1 Introduction

Markets for sovereign debt of emerging economies have developed rapidly over the past few decades. Associated with the enormous growth of sovereign debt markets have been the recurrent large-scale sovereign debt crises.¹ To resolve debt crises in the absence of an international government bankruptcy law, the defaulting countries and lenders usually renegotiate over the reduction of defaulted debt.² Despite the importance of post-default debt renegotiation to sovereign borrowing and default, the existing literature does not contain a model that adequately captures the strategic considerations at play in the international capital markets.

In a pioneering work on sovereign debt, Eaton and Gersovitz (1981) argue that a country's incentive to make repayments is to preserve its future access to international credit markets. The sovereign debt literature also emphasizes the role of direct sanctions in repayment enforcement, as first pointed out by Bulow and Rogoff (1989a). However, all the papers in this literature assume that a country either fully repays its debt or defaults completely, incurring the default penalties. The manner in which a debt crisis is resolved plays no role in the default decision. Regarding sovereign debt renegotiation, Bulow and Rogoff (1989b) present a model in which direct sanctions are avoided. Fernandez and Rosenthal (1990) analyze debt renegotiation through which the borrowing country gains improved future access to capital markets. Yet, the dynamic bargaining games analyzed in these papers are embedded in a static borrowing model. Therefore, a country's consideration for its future borrowing plays no role in the renegotiation.

This paper takes the challenge to incorporate both sovereign default and debt renegotiation into a dynamic equilibrium model. We develop a small open economy model to investigate the connection between default, debt renegotiation, and interest rates in a dynamic borrowing framework. The model features both endogenous default risk and endogenous debt recovery rates. With this model, we theoretically and quantitatively study the determination of debt recovery rates and how debt renegotiation interacts with a country's default decision. Moreover, in a quantitative exercise we analyze the valuation of sovereign bonds and map the model to Argentine data.

In the model, a risk-averse country and risk-neutral competitive financial intermediaries

¹There are 84 events of sovereign default from 1975 to 2002 according to Standard and Poor's. The largest in history is the Argentine debt crisis on international bonds of over \$82 billion in 2001.

²See Chuhan and Sturzenegger (2003) for a description of sovereign debt renegotiations between 1980 and 2000. Argentine sovereign debt restructuring was completed in 2005, and about 70% defaulted debt was reduced.

trade one-period discount bonds. Financial intermediaries who have credit relationship with the country can coordinate their actions. The country faces stochastic endowments and has an option to default. Default may result in the loss of future access to capital markets, or lead to direct sanctions imposed by the lenders. However, through renegotiation over debt reduction, the inefficient sanctions can be lifted and the defaulting country can restore its reputation, regaining access to capital markets once the renegotiated debt is repaid in full. In the meantime, the lenders can recover a part of the defaulted debt. Debt recovery rates, which are endogenously determined in a Nash bargaining game, affect a country's ex ante incentive to default. In equilibrium, sovereign bonds are priced to compensate the lenders for both the risk of default and the risk of debt restructuring.

We first establish the existence of a recursive equilibrium in the model economy. We analytically characterize the equilibrium bond prices and equilibrium debt recovery schedule. First, because the total renegotiation surplus is independent of the level of defaulted debt, we show that the debt recovery rates decrease with indebtedness, and there is no debt reduction for small-scale debt default. We also find that default may arise in equilibrium, and a country is more likely to default if it has a higher level of debt. Finally, interest rates increase with the level of debt due to the higher default probability and lower debt recovery rate.

We then show that the model can account for the dynamics of sovereign bond spreads in emerging economies. Neumeyer and Perri (2005) and Uribe and Yue (2006) document the countercyclical country interest rates for emerging markets. They show that countercyclicality of sovereign bond spreads exacerbates the business cycle fluctuations in these countries. We use the model to analyze quantitatively the sovereign debt of Argentina from 1994 to 2001. The model generates the countercyclicality of bond spreads. In the model, when a country gets a bad shock, the expected debt recovery rate is smaller according to debt renegotiation. Thus default decision helps the country to better share the income risk. Thus default risk is higher, and correspondingly the sovereign bond spreads are higher in recessions. Because the model introduces both endogenous default risk and endogenous debt recovery rate, the model also successfully accounts for the high volatility of the Argentine bond spreads, which has been difficult to match in previous work. We further show that the model can replicate the time series of Argentine bond spreads from 1994 to 2001. In addition, we quantitatively examine the role of debt renegotiation in explaining the stylized facts related to sovereign default and bond spreads. We demonstrate that the changes in bargaining power have a great impact on debt recovery rates and bond spreads as well as on the sovereign borrowing.

This paper takes incomplete asset markets as given like in Zame (1993), and we analyze the consequences of debt default and renegotiation. Our model is thus distinguished from the study of optimal contract with the lack of commitment, such as Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000), Kehoe and Perri (2002). In these works, perfect risk sharing is not achieved due to the commitment problem even though the asset markets are complete. Yet default does not arise in equilibrium, and the incentive to default is higher in good states. Because of incomplete market structure, our model generates default in equilibrium and positive default risk premium, which can be higher in bad states. Some recent works that share this feature are Chatterjee, Corbae, Nakajima and Rios-Rull (2002) on consumer credit default and Arellano (2006) and Aguiar and Gopinath (2006) on sovereign default. These studies assume a zero debt recovery rate and exogenous resumption of credit relationship. In contrast, our paper endogenizes debt recovery rates and the length of financial exclusion after default by studying debt renegotiation. We show that the endogenous debt renegotiation is important to quantitatively accounts for the dynamics of sovereign bond spreads. Another related work is by Kovrijnykh and Szentes (2005). They characterize the endogenous transition between competitive international credit market and monopolistic credit market structure which excludes defaulting countries from getting new debt. Our paper studies a dynamic model with default and debt renegotiation under different market structures, and explicitly show the outcome of debt reduction and its impact on *ex ante* default and debt pricing.

Debt renegotiation in our model generates endogenous default penalty, which in turn affects a country's *ex ante* incentive to default. This modelling feature is related to several papers in the optimal contract literature, such as Phelan (1995), Krueger and Uhlig (2004) and Cooley, Marimon and Quadrini (2004). They endogenize an agent's outside options by assuming that defaulting agents can start a new credit relationship with a competing principal. This paper is related to these works in endogenizing the value of default. The difference is that our model uses incomplete markets and generates default and debt reduction in equilibrium. Thus the model can address many important issues in the context of international borrowing and lending.

The remainder of the paper is organized as follows. Section 2 describes the model environment. Section 3 presents the sovereign borrower and lenders' problems and defines a recursive equilibrium. We then demonstrate the existence of a recursive equilibrium and characterize the equilibrium bond prices and debt recovery rates. Section 4 provides the model calibration and the results of the quantitative analysis. We conduct sensitivity analysis and additional experiments in Section 5. Finally, Section 6 offers concluding remarks.

The proofs and computation algorithm are in the Appendix.

2 The Model Environment

We study sovereign default and debt renegotiation in a dynamic model of a small open economy. We consider a risk-averse country that cannot affect the world risk free interest rate. The country's preference is given by the following utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

where $0 < \beta < 1$ is the discount factor, c_t denotes the consumption in period t and $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the period utility function, which is continuous, strictly increasing, strictly concave, and satisfies the Inada conditions. The discount factor reflects both pure time preference and the probability that the current sovereignty will survive into the next period.³ In each period the country receives an exogenous endowment of the single non-storable consumption good y_t . The endowment y_t is stochastic, drawn from a compact set $Y = [\underline{y}, \bar{y}] \subset \mathbb{R}_+$. $\mu_y(y_t|y_{t-1})$ is the probability distribution function of a shock y_t conditional on the previous realization y_{t-1} .

International financial intermediaries are risk-neutral and have perfect information on the country's endowment and asset position. We also assume that they can borrow or lend as much as needed at a constant world risk-free interest rate r on the international capital markets.

Capital markets are incomplete. The country and financial intermediaries can borrow or lend only via one-period zero-coupon bonds. The face value of a discount bond is denoted as b' , specifying the amount to be repaid next period. When the country purchases bonds, $b' > 0$, and when it issues new bonds, $b' < 0$. The set of bond face values is $B = [b_{\min}, b_{\max}] \subset R$, where $b_{\min} \leq 0 \leq b_{\max}$. We set the lower bound $b_{\min} < -\frac{\bar{y}}{r}$, which is the largest debt level that the country could repay. The upper bound b_{\max} is the highest level of assets that the country may accumulate.⁴ Let $q(b', y)$ be the price of a bond with face value b' issued by the country with an endowment shock y . The bond price function will be determined in equilibrium.

³Grossman and Van Huyck (1988) construct a model of sovereign borrowing where the time discount factor is a product of time preference coefficient and the government's survival probability of staying in power next period.

⁴ b_{\max} exists when the interest rates on a country's saving are sufficiently small compared to the discount factor, which is satisfied in our paper as $r\beta < 1$.

We assume that the financial intermediaries always commit to repay their debt. But the country is free to decide whether to repay its debt or to default. We denote the country's default history by a discrete variable $h \in \{0, 1\}$. Let $h = 0$ stand for no unresolved default on the country's record, that is, a good credit record; whereas $h = 1$ indicates an unresolved sovereign default in the country's credit history, or a bad credit record.

If a country with a good credit record ($h = 0$) defaults on its debt $b < 0$, the present value of its debt is reduced to a fraction $\alpha(b, y)$, which is the debt recovery rate determined in debt renegotiation. The defaulting country does not pay anything this period and has unpaid debt arrear $\alpha(b, y)b$ next period. However, the country's credit record deteriorates in the next period ($h' = 1$).

If a country has a bad credit record ($h = 1$) and unpaid debt arrear $b < 0$, the country has unresolved default. The country is then subject to exclusion from financial markets and direct output cost, both of which are empirically relevant. First, default incurs the reputation cost of financial exclusion, and there is no saving opportunity after default, as in Eaton and Gersovitz (1981) and other reputation models of sovereign debt.⁵ We make this assumption so that the model can support positive amount of borrowing even without direct output loss. The country may also face a direct output loss that is equal to a fraction λ_d of endowment, $0 \leq \lambda_d \leq 1$.⁶ In this case, the defaulting country can restore its good credit record by repaying the debt arrear. As in Fernandez and Rosenthal (1990) and Cole, Dow and English (1995), we assume that once the debt arrear is cleared, the country's credit record is upgraded and it regains its access to capital markets. Thus, the resumption of the international credit relationship is endogenous, depending on the amount of the debt arrear and the country's economic condition.

When default occurs, the lenders take collective action and bargain with the country over debt reduction in a Nash bargaining game. The debt recovery rate $\alpha(b, y)$ is determined in the post-default renegotiation and depends on the defaulted debt value b and endowment shock y .⁷ In the model, upon the bargaining agreement, the present value of defaulted debt is reduced to a fraction $\alpha(b, y)$ of the unpaid debt b . Should the renegotiation fail,

⁵It is well known that reputation models are subject to the Bulow and Rogoff (1989a) critique, which says that pure reputation mechanism cannot support positive international debt if defaulting countries can save at the same interest rate. Cole and Kehoe (1998), Kletzer and Wright (2000), Wright (2002) and Amador (2003) analyze various ways to reduce the range of a defaulting country's saving mechanism so that reputation model can generate positive international lending.

⁶The direct output loss is observed empirically. We takes the proportional loss specification for simplicity. All the theoretical analysis goes through even when there is not output loss, e.g. $\lambda_d = 0$.

⁷Kovrijnykh and Szentes (2005) present a model in which lenders are competitive in pre-default periods and become monopolists after default.

international lenders would lose their investment. The country would be excluded from the financial markets forever from that period on.

The timing of decisions within one period is summarized in Figure 1. At the beginning of one period, an endowment shock y is realized, and the country has assets b . When the country has a good credit record ($h = 0$), it decides to repay its debt or default. If the country decides not to default, then it chooses b' and its credit record remains good for the next period ($h' = 0$). If the country chooses to default, debt renegotiation takes place. The country's debt is reduced to $\alpha(b, y)b$. The credit record deteriorates ($h' = 1$). When the country starts with a bad credit record ($h = 1$) with unpaid debt arrear b , it determines how much debt to repay. If some debt arrear is not repaid ($b' > 0$), then the country's credit record remains bad ($h' = 1$). Once all the debt arrear is repaid ($b' = 0$), the country regains its good credit record ($h' = 0$).

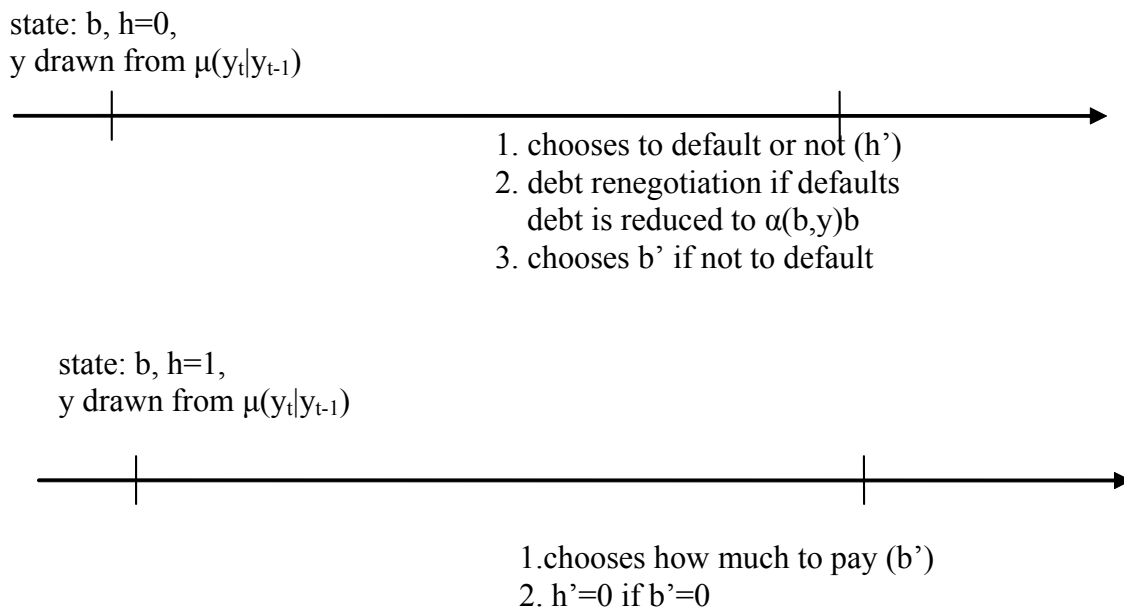


Figure 1: Timeline of the Model

3 Recursive Equilibrium

In this section, we define and characterize a dynamic recursive equilibrium.

3.1 Sovereign Country's Problem

The country's objective is to maximize the expected lifetime utility. The country makes its default decision and determines its assets for next period, given the current asset position b and the endowment shock y . Let $v(b, h, y) : \mathcal{L} \rightarrow \mathcal{R}$ be the life-time value function for the country that starts the current period with the credit record h , asset position b , and endowment shock y , where $\mathcal{L} = B \times \{0\} \times Y \cup B_- \times \{1\} \times Y$.⁸ We restrict the space of bond price functions to be $Q = \{q|q(b, y) : B \times Y \rightarrow [0, \frac{1}{1+r}]\}$, and the space of debt recovery schedules to be $A = \{\alpha|\alpha(b, y) : B_- \times Y \rightarrow [0, 1]\}$. Given any bond price function $q \in Q$ and debt recovery schedule $\alpha \in A$, the country solves its optimization problem.

For $b \geq 0$ and $h = 0$, the country has a good credit record and savings. The country receives payments from the financial intermediaries and determines its next-period asset position b' to maximize utility. Thus, the value function is

$$v(b, 0, y) = \max_{c, b' \in B: c + q(b', y)b' = y + b} u(c) + \beta \int_Y v(b', 0, y') d\mu(y'|y) \quad (2)$$

For $b < 0$ and $h = 0$, the country has a good credit record and outstanding debt. If the country honors its debt obligation, it chooses its next-period asset position b' and consumes. If the country defaults, it cannot borrow or save in the current period. Moreover, the country's credit record deteriorates to $h' = 1$. But the country gets its debt reduced to $\alpha(b, y)b$, and its next period debt position is $\alpha(b, y)b(1+r)$. The country determines to default or not optimally. Its optimal value function is:

$$v(b, 0, y) = \max \{v^r(b, 0, y), v^d(b, 0, y)\} \quad (3)$$

$v^r(b, 0, y)$ is the value function if the country does not default:

$$v^r(b, 0, y) = \max_{c, b' \in B: c + q(b', y)b' = y + b} u(c) + \beta \int_Y v(b', 0, y') d\mu(y'|y)$$

$v^d(b, 0, y)$ is the value of default:

$$v^d(b, 0, y) = u(y) + \beta \int_Y v(\alpha(b, y)b(1+r), 1, y') d\mu(y'|y)$$

where $v(\cdot, 1, y')$ is defined subsequently.

For $h = 1$, the country has a bad credit record and unpaid debt arrear $b < 0$. The

⁸Note that only the country with a good credit record can have savings,

country is excluded from financial markets, and its endowment suffers a proportional loss of $\lambda_d y$. The country chooses optimally the fraction of the debt arrear to be paid back. We assume that the creditor can enforce the payment of interests that accrued with the unpaid debt.⁹ If the country repays partially, its next-period credit record remains bad. The debt arrear rolls over at the interest rate r . The value function is thus

$$v(b, 1, y) = \max_{c, b' \in [b, 0]: c + \frac{b'}{1+r} = (1-\lambda_d)y + b} u(c) + \beta \int_Y v(b', 1, y') d\mu(y'|y) \quad (4)$$

Lastly, when all the debt arrear is paid, the country regains its full access to the markets. That is for $b = 0$ and $h = 1$, the value function is:

$$v(0, 1, y) = v(0, 0, y) \quad (5)$$

The country's default policy can be characterized by default sets $D(b) \subset Y$. Default set is the set of endowment shock y 's for which default is optimal given the debt position b .

$$D(b) = \{y \in Y : v^r(b, 0, y) \leq v^d(b, 0, y)\}$$

In the model economy, the country may have an incentive to default because the default option and debt renegotiation introduce contingencies into non-contingent sovereign debt contracts and facilitate interstate consumption smoothing. But intertemporal consumption smoothing is hurt due to higher interest rates and more limited access to the financial markets after default.

3.2 The Debt Renegotiation Problem

The debt renegotiation takes the form of a generalized Nash bargaining game. In the model, upon the bargaining agreement, the present value of defaulted debt is reduced to a fraction $\alpha(b, y)$ of the unpaid debt b . The value of such an agreement to the country is $v^d(b, 0, y) = u(y) + \beta \int_Y v(\alpha(b, y)(1+r)b, 1, y') d\mu(y'|y)$, which is the expected life-time utility of defaulting when the debt recovery rate is $\alpha(b, y)$. This value takes into account the impact of debt reduction as well as a temporary debt exclusion associated with a bad credit record. The lenders gets the present value of the reduced debt $\alpha(b, y)b$.

We assume that the renegotiation takes place only once for one default event. The

⁹This is a technical assumption to ensure that the state space for debt arrear not to go unbounded. This assumption also implies that the creditor get all the reduced debt with certainty. Therefore, the market interest rate for debt arrear is the risk free rate r .

threat point of the bargaining game is that the country stays in permanent autarky and the creditors get nothing. Moreover, lenders could impose direct sanctions $\lambda_s y$ on the country, which is in addition to defaulting country's direct output loss $\lambda_d y$.¹⁰ The expected value of autarky to the country, $v^{aut}(y)$, is given in a recursive form.

$$v^{aut}(y) = u((1 - (\lambda_d + \lambda_s))y) + \beta \int_Y v^{aut}(y') d\mu(y'|y) \quad (6)$$

One remark is that the one-round bargaining assumption keeps the model tractable as there is no need to track the rounds of bargaining or the debt arrear based different reduction schedules. And our model captures well the interaction between default and renegotiation in a general model, because further rounds of renegotiation play similar role in affecting the country's ex ante default incentive.¹¹

For any debt recovery rate a , we denote the country's surplus in the Nash bargaining by $\Delta^B(a; b, y)$, which is the difference between the value of accepting the debt recovery rate a and the value of rejecting it, given the country's debt level b and endowment y .

$$\Delta^B(a; b, y) = \left[u(y) + \beta \int_Y v(a(1+r)b, 1, y') d\mu(y'|y) \right] - v^{aut}(y) \quad (7)$$

The surplus to the country comes from two sources. First, although the country's credit record becomes bad in the next period, the expected length of financial exclusion is finite. Thus, the defaulting country gains from losing the access to capital markets for a temporary periods rather than permanently. Second, the direct output loss is smaller under the renegotiation agreement because no sanctions are imposed.

The surplus to the risk-neutral lender is the present value of recovered debt.

$$\Delta^L(a; b, y) = -ab \quad (8)$$

If lenders have all the bargaining power, then they could extract debt repayments up to the full amount of a country's cost of default. If, on the other hand, the borrowing country can make take-it-or-leave-it offers, then it gets a complete debt reduction in the bargaining. To analyze the general case, we assume that the borrower has a bargaining power θ and the

¹⁰There is direct sanction because creditors may litigate in foreign courts or apply trade sanctions, as discussed in Bulow and Rogoff (1989b). The direct sanction assumption is not needed to study the model's theoretical properties. Yet it improves the model's fit in the quantitative analysis.

¹¹A generalization is to allow for multiple rounds of costly renegotiations. Allowing for continuous costless renegotiation generates either risk-free debt or no international lending.

lender has a bargaining power $(1 - \theta)$. The bargaining power parameter θ summarizes the institutional arrangement of debt renegotiation. To ensure that the bargaining problem is well defined, we define the bargaining power set $\Theta \subset [0, 1]$ such that for $\theta \in \Theta$ the renegotiation surplus has a unique optimum for any debt position b and endowment shock y .

Given debt level b and endowment y , the debt recovery rate $\alpha(b, y) \in A$ solves the following bargaining problem:

$$\begin{aligned} \alpha(b, y) &= \arg \max_{a \in [0, 1]} \left[(\Delta^B(a; b, y))^\theta (\Delta^L(a; b, y))^{1-\theta} \right] \\ \text{s.t. } \Delta^B(a; b, y) &\geq 0 \\ \Delta^L(a; b, y) &\geq 0 \end{aligned} \quad (9)$$

Because the debt recovery schedule that maximizes the total renegotiation surplus is conditional on the country's endowment shock and debt position, the renegotiation outcome provides better insurance to the country if it decides to default.

3.3 International Financial Intermediaries' Problem

Taking the bond price function as given, the financial intermediaries choose the amount of debt b' to maximize their expected profit π . Their expected profit is given by

$$\pi(b', y) = \begin{cases} q(b', y) b' - \frac{1}{1+r} b' & \text{if } b' \geq 0 \\ \frac{[1-p(b', y)+p(b', y)\gamma(b', y)]}{1+r} (-b') - q(b', y) (-b') & \text{if } b' < 0 \end{cases} \quad (10)$$

where $p(b', y)$ is the expected probability of default for a country with an endowment y and indebtedness b' , and $\gamma(b', y)$ is the expected recovery rate, given by the expected proportion of defaulted debt that the creditors can recover, conditional on default.

Because we assume that the market for new sovereign debt is completely competitive, the financial intermediaries' expected profit is zero in equilibrium. Using the zero expected profit condition, we get

$$q(b', y) = \begin{cases} \frac{1}{1+r} & \text{if } b' \geq 0 \\ \frac{[1-p(b', y)]}{1+r} + \frac{p(b', y)\gamma(b', y)}{(1+r)} & \text{if } b' < 0 \end{cases} \quad (11)$$

When the country lends to the intermediaries, $b' \geq 0$, the sovereign bond price is equal to the price of a risk-free bond $\frac{1}{1+r}$. When the country borrows from the intermediaries,

$b' < 0$, there exist the risks of default and debt restructuring. The sovereign bond is priced to compensate the financial intermediaries for bearing both risks.¹²

Equation (11) shows that default risk has the first-order effect on bond price as it enters the first term of pricing function in the linear form. The debt recovery rate affects the sovereign bond prices through its combined effect with default risk. Thus it does not have direct first order effect on the level of bond prices. Yet, debt recovery rate has a significant effect on the second moment of sovereign bond prices and an important indirect impact on ex ante default risk, as the subsequent section shows.

Since $0 \leq p(b', y) \leq 1$ and $0 \leq \gamma(b', y) \leq 1$, the bond price $q(b', y)$ lies in $[0, \frac{1}{1+r}]$. The interest rate on sovereign bonds, $r^s(b', y) = \frac{1}{q(b', y)} - 1$, is bounded below by the risk-free rate. The difference between the country interest rate and the risk free rate is the country's credit spread, $s(b', y) = r^s(b', y) - r$.

3.4 Recursive Equilibrium

We now define a stationary recursive equilibrium in the model economy.

Definition 1 *A recursive equilibrium is a set of functions for (i) the country's value function $v^*(b, h, y)$, asset holdings $b^*(b, h, y)$, default set $D^*(b)$, consumption $c^*(b, h, y)$ (ii) recovery rate $\alpha^*(b, y)$ and (iii) pricing function $q^*(b', y)$ such that*

1. *Given the bond price function $q^*(b', y)$ and debt recovery rate $\alpha^*(b, y)$, the value function $v^*(b, h, y)$, asset holding $b^*(b, h, y)$, consumption $c^*(b, h, y)$ and default set $D^*(b)$ satisfy the country's optimization problem (2), (3), and (4).*

2. *Given the bond price function $q^*(b', y)$ and value function $v^*(b, h, y)$, the recovery rate $\alpha^*(b, y)$ solves the debt renegotiation problem (9).*

3. *Given the recovery rate $\alpha^*(b, y')$, the bond price function $q^*(b', y)$ satisfies the zero expected profit condition for intermediaries (11), where the default probability $p^*(b', y)$ and expected recovery rate $\gamma^*(b', y)$ are consistent with the country's default policy and renegotiation agreement.*

In equilibrium, the default probability $p^*(b', y)$ is related to the country's default policy in the following way:

$$p^*(b', y) = \int_{D^*(b')} d\mu(y'|y) \tag{12}$$

¹²The price functions for consumer debt in Chatterjee et al. (2002) and sovereign debt in Arellano (2006) are our model's special cases in which debt recovery rate is zero, and the default risk alone determines the bond price.

The expected recovery rate $\gamma^*(b, y)$ in equilibrium is determined by

$$\begin{aligned}\gamma^*(b', y) &= \frac{\int_{D^*(b')} \alpha^*(b', y') d\mu(y'|y)}{\int_{D^*(b')} d\mu(y'|y)} \\ &= \frac{\int_{D^*(b')} \alpha^*(b', y') d\mu(y'|y)}{p^*(b', y)}\end{aligned}\tag{13}$$

The numerator is the expected proportion of debt that the investors can recover, and the denominator is the default probability.

We can establish the existence of a recursive equilibrium in the model economy as stated in Theorem 1.¹³

Theorem 1 *Given any bargaining power $\theta \in \Theta$, a recursive equilibrium exists.*

Proof. See Appendix. ■

3.5 Characterization of a Recursive Equilibrium

We now proceed to establish some properties of a recursive equilibrium.

Theorem 2 *For a bargaining power $\theta \in \Theta$, there exists a threshold $\bar{b}(y) \leq 0$ such that the equilibrium debt recovery function α satisfies*

$$\alpha^*(b, y) = \begin{cases} \frac{\bar{b}(y)}{b} & \text{if } b \leq \bar{b}(y) \\ 1 & \text{if } b \geq \bar{b}(y) \end{cases}$$

Proof. See Appendix. ■

The intuition for Theorem 2 is the following: After default, bygones are bygones. The defaulting country cares about the total amount of reduced debt, which affects the expected duration of financial exclusion. At the same time, the financial intermediaries are solely concerned with the total recovery on defaulted debt. Therefore, the bargaining on debt recovery rate is equivalent to the renegotiation over the reduced debt. There is an optimal value of reduced debt that maximizes total renegotiation surplus. Hence, debt recovery rates decrease inversely with the amount of defaulted debt, and there is no debt reduction for debt levels smaller than the threshold.

¹³We cannot prove the uniqueness of the recursive equilibrium. But we do not find multiple equilibria in the numerical computation.

Country	Pakistan	Ukraine	Ecuador	Russia	Argentina
Time of default	98.12	98.09	99.08	98.11	01.11
Defaulted debt (billion \$)	0.75	2.7	6.6	73	82.3
Defaulted debt/output	0.10	0.09	0.46	0.12	0.32
Defaulted debt/reserves	0.37	1.82	3.54	2.65	5.64
Debt recovery rate	100%	100%	40%	36.5%	28%

Note: Data are from World Bank, Chuhan and Sturzenegger (2003), and Moody's (2003).

Table 1: Statistics for Different Bargaining Powers

Despite the limited data availability on sovereign debt renegotiation, some observations from the recent sovereign bond exchanges are largely consistent with Theorem 2's prediction. Table 1 shows the scale of debt crises and debt recovery rates for Ukraine, Pakistan, Ecuador, Russia, and Argentina. The debt recovery rate is lower for a more severe debt crisis, both in terms of the dollar amount and relative to the country's output and foreign reserves. Thus the model prediction is in line with the empirical observations.

The country's *ex ante* incentive to default depends on the *ex post* renegotiation agreement on debt reduction in equilibrium. Given the equilibrium debt recovery schedule $\alpha(b, y)$, characterized by Theorem 2, and the endowment shock y , the value function of a defaulting country is independent of the level of debt if it is larger than $\bar{b}(y)$. Therefore, we can show that the default set increases with the country's indebtedness and the equilibrium default probability increases with the level of debt.

Theorem 3 *Given an equilibrium debt recovery schedule $\alpha^*(b, y)$ and an endowment $y \in Y$, for $b^0 < b^1 \leq \bar{b}(y)$, if default is optimal for b^1 , then default is also optimal for b^0 . That is $\bar{D}^*(b^1) \subseteq \bar{D}^*(b^0)$.*

Proof. See Appendix. ■

Theorem 4 *Given an equilibrium debt recovery schedule $\alpha^*(b, y)$ and an endowment $y \in Y$, the country's probability of default in equilibrium satisfies $p^*(b^0, y) \geq p^*(b^1, y)$, for $b^0 < b^1 \leq \bar{b}(y) \leq 0$.*

Proof. See Appendix. ■

Given the endogenous debt recovery rates, our model predicts that default probability increases with the level of debt, as in Eaton and Gersovitz (1981). The key difference is that they assume a zero debt recovery rate and rule out the possibility of debt renegotiation, yet our model obtains this result in a more general set up with endogenous debt renegotiation.

We also characterize the equilibrium bond price schedule.

Theorem 5 *Given an endowment $y \in Y$, for $b^0 < b^1 \leq \bar{b}(y) \leq 0$, an equilibrium bond price $q^*(b^0, y) \leq q^*(b^1, y)$.*

Proof. See Appendix. ■

In equilibrium, bond prices depend on both the risk of default and the expected debt recovery rates. For a high level of debt, the default probability is higher, but the expected debt recovery rate is lower. Therefore, equilibrium bond prices decrease with indebtedness. This result is consistent with the empirical evidence, for example, Edwards (1984).

The next theorem characterizes the debt arrear repayment policy of a defaulting country.

Theorem 6 *Given an endowment $y \in Y$, if there exists a level of debt $\tilde{b} < 0$ that satisfies*

$$\begin{aligned} & \sup_{b' < 0} \left((1 - \lambda_d) y + \tilde{b} - \frac{b'}{1+r} \right) + \beta \int_Y v(b', 1, y') d\mu(y'|y) \\ &= u \left((1 - \lambda_d) y + \tilde{b} \right) + \beta \int_Y v(0, 0, y') d\mu(y'|y) \end{aligned} \quad (14)$$

then for all $b \in B_-$ and $b > \tilde{b}$, it is strictly optimal for the defaulting country to repay its debt arrear in full, and for all $b \in B_-$ and $b < \tilde{b}$, a partial repayment is strictly optimal.

Proof. See Appendix. ■

This theorem implies that if the country fully repays the debt arrear b and regains access to financial markets next period, then it also chooses to do so with a lower level of debt arrear. If the country decides not to repay the debt in full, it will do the same for higher debt arrear.

Note that although there is no delay in debt renegotiation due to the Nash bargaining model set up, the model generates endogenous exclusion from financial markets after default, which is closely related to the debt renegotiation outcome. Based on the above theorem, the expected duration of financial exclusion increases with the amount of reduced debt after default and debt arrear in general. This is thus complementary to Kovrijnykh and Szentes (2005) who find that countries with debt overhang are more likely to reaccess financial markets after a series of good shocks. Moreover, subsequent quantitative analysis shows that our model also generates a shorter financial exclusion if a series of good shocks realize.

4 Quantitative Analysis

In this section, we calibrate the model to analyze quantitatively the sovereign debt of Argentina.

4.1 Calibration

We define one period as a quarter. The utility function for the country has constant relative risk-aversion (CRRA). So

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \quad (15)$$

where σ is the coefficient of risk aversion. We set the risk aversion coefficient to 2, which is standard in the macroeconomics literature. We set the risk-free interest rate r to 1%, the average quarterly interest rates on 3 month US treasury bills. The output loss parameter λ_d is set to 2%, which is in line with the output contraction estimated by Sturzenegger (2002).

The endowment process is calibrated to the Argentine quarterly output from 1980Q1 to 2003Q4, which are seasonally adjusted real GDP data from the Ministry of Finance (MECON). To capture the stochastic trend in GDP, we model the output growth rate as an AR(1) process:

$$\log g_t = (1 - \rho_g) \log(1 + \mu_g) + \rho_g \log g_{t-1} + \varepsilon_t^g \quad (16)$$

where growth rate is $g_t = \frac{y_t}{y_{t-1}}$, growth shock is $\varepsilon_t^g \stackrel{iid}{\sim} N(0, \sigma_g^2)$, and $\log(1 + \mu_g)$ is the expected log gross growth rate of the country's endowment. We estimate the endowment process to match the average growth rate, as well as the standard deviation and autocorrelation of HP detrended output.

Because a realization of the growth shock permanently affects endowment, the model economy is nonstationary. In the quantitative analysis, we detrend the model by the lagged endowment level y_{t-1} . The detrended counterpart of any variable x_t is thus $\hat{x}_t = \frac{x_t}{y_{t-1}}$. Equilibrium value function, bond price function and debt recovery schedule are also re-defined using the detrended variables.¹⁴

In the last part of the calibration, we pick the time discount factor β , the country's bargaining power θ , and direct sanctions parameter λ_s to match the average default fre-

¹⁴The theoretical analysis in preceding sections is conducted with a stationary endowment. It is straight forward to transform the model with nonstationary endowment to a detrended problem, and the analysis goes through. The details are in a technical note available upon request.

Statistics (quarterly)	Source	Data	Model
Panel A			
World risk-free interest rate	US Treasury-bill interest rates	1%	1%
Output loss in debt crises	Sturzenegger (2002)	2%	2%
Panel B			
Average output growth rate	MECON	0.420	0.42%
Output standard deviation	MECON	4.35%	4.35%
Output autocorrelation	MECON	0.82	0.82
Panel C			
Average debt service/output	the World Bank	9.54%	9.69%
Default frequency	Reinhart et al. (2003)	0.69%	0.54%
Average recovery rate	Moody's (2003)	28%	28%

Table 2: Target Statistics for Argentina (1980.1-2003.4)

quency, average debt recovery rate, and debt service-to-output ratio of Argentina, using a simulated method of moments (SMM). Reinhart, Rogoff and Savastano (2003) report four episodes of sovereign defaults in Argentina's external debt from 1824 to 1999. In 2001, Argentina defaulted a fifth time on its external debts, making its average default frequency 2.78% annually or 0.69% quarterly. The average debt recovery rate is estimated as the first available bid price for defaulted bonds 30 days after Argentine default in 2001. According to the Moody's (2003) report, Argentina's average recovery rate is 28%.¹⁵ The debt service-to-output ratio includes both short term debt and debt service on long term debt, which is calculated using data from the World Bank.¹⁶ For the period 1980-2003, the ratio of Argentina's debt service to its gross national income is 9.54%.

The time discount factor β is found to be 0.74. This high degree of impatience helps to generate frequent defaults. It also reflects the high political instability in Argentina with 14 presidents from 1981 to 2004. The country's quarterly survival probability is 84.7%, which makes the value of pure time discount factor 0.873. The sanctions parameter is 1.22%, which shows that the creditors have some power to impose direct sanctions on the country. Finally, the bargaining power is 0.83, which shows that Argentina has a more favorable position in debt renegotiation than the international investors. Table 2 presents the statistics for Argentina that we use as the calibration target. Table 3 summarizes the

¹⁵In the recent bond exchanges for the Argentine defaulted debt, the recovery rate is 30%. According to Moody's (2003), the average recovery rate is 41% for sovereign borrowers.

¹⁶We calibrate the model to debt service-to-GDP ratio in stead of debt stock-to-GDP ratio because debt stock (including long-term debt) is the total discounted value of debt service over future years. Our model studies quarterly debt. Thus, debt service-to-GDP ratio is the appropriate debt index for calibration.

Parameter	Symbol	Value
Panel A		
Coefficient of Risk Aversion	σ	2
Risk Free Interest Rate	r	1%
Output Loss in Default	λ_d	2%
Panel B		
Average Endowment Growth	μ_g	0.42%
Std Dev. to Endowment Growth Shock	σ_g	2.53%
Endowment Growth AR(1) coefficient	ρ_g	0.41
Panel C		
Time Discount Factor	β	0.74
Sanction Threat	λ_s	1.22%
Bargaining Power	θ	0.83

Table 3: Model Parameter Values in the Model

calibration results.

4.2 Comparison of Model to Data

We show the properties of the equilibrium in the calibrated model first. Then, we compare the results of the model simulation to the data.

Figure 2 plots the equilibrium debt recovery schedule. As Theorem 2 shows, if a country defaults with a small amount of debt, there is no debt reduction. As the amount of defaulted debt increases, the debt recovery rate decreases. In addition, the numerical results show that debt reduction threshold $\bar{b}(y)$ is a decreasing function of the economic shock. Hence, the debt recovery rate is higher for a defaulting country with a good economic shock, and vice versa. Therefore, debt renegotiation allows default to better complete markets because defaults in good times end up with higher level of debts than defaults in bad times do. Default is a more valuable option in this case compared to the model with constant debt recovery rates. Such recovery rates also contribute to the countercyclicality of interest rates because the *ex ante* incentive to default is higher when *ex post* debt reduction is big in bad times.

Figure 3 plots the default probabilities. For a country with a very low level of debt, there is virtually no default, regardless of the endowment shock. And default probability is zero for a range of debt level beyond the debt reduction threshold. This interesting result shows that the lowly-indebted country may choose not to default even when the debt renegotiation

can generate positive debt reduction because the cost of financial exclusion is higher than the value of getting debt reduction. The default probability increases with indebtedness. Furthermore, the default probability is higher for a country with a bad economic shock because default is more valuable for the country facing a bad shock.

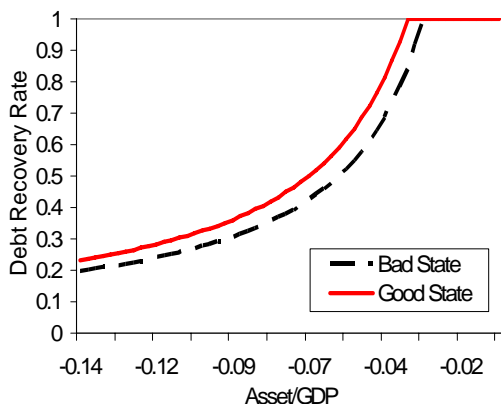


Figure 2: Recovery Rate

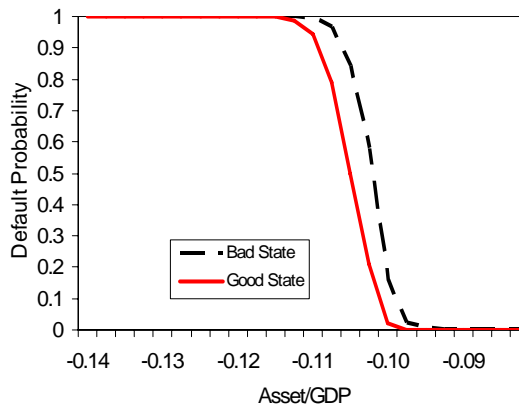


Figure 3: Default Probability

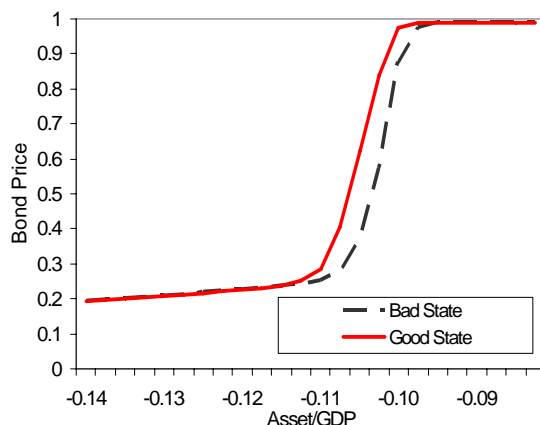


Figure 4: Bond Price Function in Benchmark Model

Figure 4 presents the bond price functions for a country with the highest and the lowest endowment shock in the current period. It shows that the bond price increases with the assets-to-output ratio (or decreases in the debt-to-output ratio), as predicted by the theory. Bond price function also increases with the endowment shock, which implies the countercyclicality of interest rates. When a country receives a bad shock, the higher debt reduction increases the country's ex ante default incentive. A higher default probability and a lower debt recovery rate generate a higher sovereign bond spread, and thus a negative correlation between spreads and output.

Non-target Statistics	Data	Model
Bond Spreads Std. Dev.(annualized)	1.68%	1.32%
Correlation between Bond Spreads and Output	-0.12	-0.18
Correlation between Bond Spreads and Current Account	0.49	0.54
Correlation between Current Account and Output	-0.88	-0.14
Consumption Std. Dev./Output Std. Dev.	1.15	1.03
Average Bond Spreads (annualized)	4.08%	1.84%
Current Account Std. Dev. (annualized)	5.40	2.32

Table 4: Model Statistics for Argentina

We feed the endowment process to the model and conduct 2000 simulations. In each round, we simulate the model for 1000 periods and extract the last 50 observations to explore the behavior of the model economy in the stationary distribution. Overall, the model matches the Argentine interest rate volatility, consumption volatility, as well as the correlations between interest rates, output, consumption and current account. Table 4 compares the model statistics with the data statistics.

The bond spreads data are quarterly spreads on Argentine foreign currency denominated 3-year bonds from 1994Q2 to 2001Q4, taken from Broner, Lorenzoni and Schmukler (2004).¹⁷ The consumption and current account data are seasonally adjusted from 1980Q1 to 2003Q4, taken from MECON. Current account are calculated as a ratio of output.

The model simulation closely matches the volatility of the Argentine interest rates in the data, which has been found hard to explain in the literature. The model can account for about 78% of volatility in the 3-year bond spreads in the data. This improves the result in the previous studies on sovereign debt spreads. In our model, the bond spreads are jointly determined by the default probabilities and debt recovery rates. Therefore, allowing for debt renegotiation breaks the one-to-one matching from default probabilities to bond spreads even though lenders are risk neutral. The debt recovery rates are correlated with default probability. In particular, default probability is higher when a larger fraction of debt reduction is expected in the post-default renegotiation. Hence, the endogenous debt renegotiation amplifies the default risk and thus the volatility of bond spreads.

Second, the model delivers the relation between bond spreads, outputs and current

¹⁷The J.P. Morgan's Emerging Markets Bond Indices (EMBI) have been commonly used in the research of country spreads of emerging economies. Yet EMBI uses bonds with maturities of 7-10 years. Broner, Lorenzoni and Schmukler find that the sovereign bond spreads have a significant term structure variation. For the period of 1994Q2 to 2001Q4. The average bond spreads increase from 4.08% for 3-year bonds to 6.24% for 12-year bonds. The corresponding volatilities increase from 1.68% to 3.12%. Because our calibrated model generates interest rates for 3-month bonds, we compare the model prediction to short-term 3-year bond spreads.

account in the data. Bond spreads are negatively correlated with output and positively correlated with current account. The negative correlation between spreads and output is accounted for because the sovereign bonds have higher default risk and lower debt recovery rate in bad states. Consequently, it is more expensive for the country to borrow in bad states. Because the growth shock is persistent, the country's current account increases due to the lower borrowing when it receives a bad shocks.¹⁸ Although lower level of debt implies relative lower bond spreads, the downward shift in the bond price schedule caused by the bad shock dominates, which implies higher bond spreads. Therefore, the bond spreads are also positive correlated with current accounts. Moreover, the current account is countercyclical in the model, although the magnitude of correlation in the model is lower than in the data.

The model also generates volatile consumption at the business cycle frequency. The consumption volatility is higher than output volatility in the data, which is a common feature of emerging economies (see Neumeyer and Perri (2005)). In our model, a good endowment shock increases permanent income more than proportionally, so the country borrows to consume more, and vice versa. Therefore, consumption is more volatile than endowment in our model. The current account is not as volatile as in the data because international borrowing and lending is the only determinant of current account in the model.

The average annual bond spread is 1.84% in the simulation, which is about 45% of the average spread in the data.¹⁹ Although the spread average is lower than the data statistic, it is conceivably higher than the results in recent studies. Note that our model assumes risk neutral creditors, so the predicted bond spreads do not include risk premium, which may increase bond spreads in the data. Moreover, the term premium between 3-month bonds analyzed in our model and 3-year bonds in the data should be taken into account in the comparison.

We also show that the model can replicate the recent Argentine debt crises and the time series of Argentine bond spreads over the past 10 years. We feed the Argentine GDP growth rate into the model and compare the time series of bond spreads. Figure 5 plots the H-P detrended output, the 3 year Argentine bond spreads, and the simulated bond spreads from 1994Q2 to 2001Q4. The figure demonstrates that the model can explain

¹⁸Atkeson (1991) develop a model with limited enforcement and moral hazard to explain this pattern of capital outflow.

¹⁹In our model, the mean spread is equal to the product of average default probability and average debt reduction rate in the model. Since the default frequency in the data is 2.78% and the average debt reduction rate is 72%, the average bond spreads in the stationary distribution is about 2%.

the recent Argentinian default episode. Before a default occurs, the country faces volatile and countercyclical interest rates. When the country gets a really bad shock, the model generates a default on the country’s debt, as what we observe in Argentina in the last quarter of 2001.

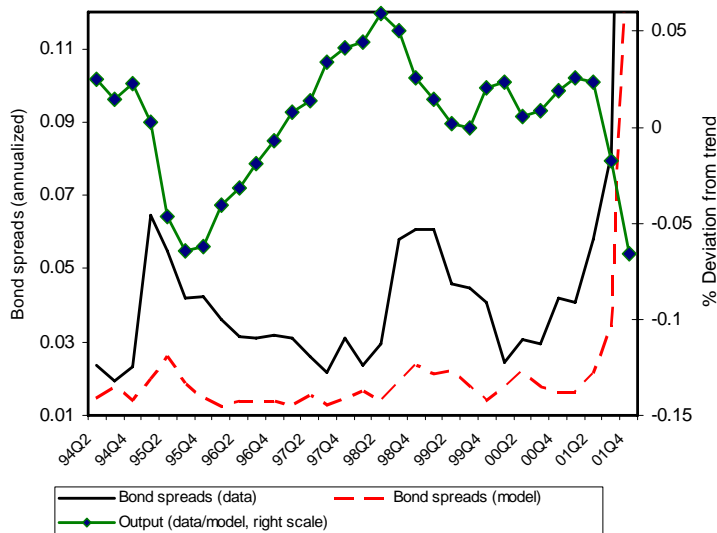


Figure 5: Output and Bond Spreads in the Data and in the Model (1994.2-2001.4)

Lastly, regarding the length of financial exclusion, the results show that the country is excluded from the financial markets for one period after default on average.²⁰ Gelos et al (2003) find that it takes less than one year for defaulted countries to regain access to international financial markets in the 1990s. Since the renegotiation agreement in the model is reached immediately after default, the financial exclusion periods in the model do not include delay in renegotiation and thus are shorter than what we observe in the data.

5 Additional Model Implications

In this section, we explore the role of endogenous recovery rates and examine how the equilibrium changes with the bargaining power. Results of sensitivity analysis are also presented.

Target Statistics	Data	Model	Comparison Model
Default Probability	2.76%	2.16%	0.76%
Average Debt Service/Output	9.54%	9.69%	10.24%
Other Statistics			
Average Recovery Rate	28%	28%	0
Average Bond Spreads	4.08%	1.84%	0.76%
Bond Spreads Std.	1.68%	1.32%	0.52%

Table 5: Role of Endogenous Renegotiation

5.1 Role of Endogenous Debt Renegotiation

We study the role of endogenous debt renegotiation by comparing our model to the one without renegotiation. In the comparison model, default leads to a full debt discharge and the defaulting country regains access to the capital markets with an exogenous return probability. Complete debt discharge corresponds to an exogenous zero debt recovery rate $\alpha = 0$ in our model. An exogenous return probability δ determines the expected length of exclusion from financial markets.

We calibrate the time discount factor and the return probability in the model without renegotiation to match the average default frequency and debt-to-output ratio. The other parameters take the same values as in benchmark model. Table 5 summarizes the results from our model and the comparison model without renegotiation.

First, the comparison model does not match the target statistics.²¹ In particular, the model without renegotiation generates significantly lower default frequency than what we see in the data. It implicates that endogenous renegotiation is needed to account for the observed high default frequency for Argentina. The comparison model also generates lower Argentine bond spreads due to a smaller default risk, despite the zero debt recovery rate assumption. Moreover, because default risk is the sole determinant of the bond price in the comparison model, and in equilibrium default is a rare event, the simulated bond spreads are less volatile. In contrast, our model with endogenous renegotiation generates higher default probability and exacerbates interest rate volatility. The main reason is that debt renegotiation brings in more state contingency in the incomplete market model. Thus the country defaults more in equilibrium. Although given default probability, the positive debt recovery rate implies lower bond spreads. The endogenous debt recovery schedule generates

²⁰We get longer periods of financial exclusion when the country has lower bargaining power.

²¹The best fitting parameters for the comparison model are 0.768 for the time discount factor, which is close to the one in our benchmark model, and 0.274 for the return probability, which implies the financial exclusion on average is 0.9 year.

bargaining power	recovery rate	$\frac{\text{debt}}{\text{output}}$	default prob.	mean s	consumption increment relative to benchmark
$\theta=0.35$	89.01%	93.14%	5.36%	1.28%	-0.356%
$\theta=0.7$	33.68%	13.46%	1.32%	0.96%	-0.034%
$\theta=0.83$	27.93%	9.69%	2.16%	1.84%	–
$\theta=0.9$	19.64%	7.83%	1.32%	1.24%	0.025%
$\theta=1$	0	2.25%	0.08%	0.12%	0.062%

Table 6: Statistics for Different Bargaining Powers

a much higher default probability, and thus can account for higher level and volatility of sovereign bond spreads.

5.2 Bargaining Power

The debt renegotiation plays a central role in our model, and the bargaining power is a key parameter that captures the bargaining protocol. Therefore, we now investigate how different bargaining powers affect the model predictions. The results are summarized in Table 6. The bargaining power parameter has a direct impact on debt recovery rate. It is intuitive that higher bargaining power for the country results in a lower debt recovery for lenders. Keeping other things fixed, the lower recovery rate increases the average bond interest rates. On the other hand, the lower debt recovery rate shifts down the bond price schedule. As a result, borrowing is discouraged and thus the debt-to-output ratio is smaller. With less borrowing, both the default probability and the bond interest rates decreases, *ceteris paribus*. Therefore, the increasing bargaining power for the country has two opposite effects on default probability and bond interest rates. How the equilibrium changes depends on which effect dominates. Table 6 shows that the default probability and average interest rates do not change monotonically with the bargaining power.

The results in Table 6 with different bargaining powers can be viewed as outcomes of policy experiments. To evaluate the impact of different policies on the country’s welfare, we calculate the consumption increment ϕ that makes the country indifferent between the economy with a certain bargaining power and the benchmark economy.²² The results show that having a higher bargaining power slightly improves the country’s welfare. When a

²²Let Λ^0 denote the ex ante utilitarian welfare in the stationary distribution for the benchmark model, and Λ^p denote the welfare in the model economy with a given bargaining power. Consumption increment ϕ is computed as $\phi = \left(\frac{\Lambda^p + 1/(1-\sigma)(1-\beta)}{\Lambda^0 + 1/(1-\sigma)(1-\beta)} \right)^{1/(1-\sigma)} - 1$. If $\phi > 0$, the country is better off with the new bargaining power than in the benchmark case. The converse also holds.

	Default prob.	Recovery rate	debt/output	Mean s	Std.(s)
Time discount factor					
$\beta = 0.74$	2.16%	27.93%	9.69%	1.84%	1.32%
$\beta = 0.8$	1.33%	34.15%	9.82%	0.94%	0.46%
$\beta = 0.9$	0.40%	47.95%	11.77%	0.20%	0.12%
Risk free rate					
$r = 0.01$	2.16%	27.93%	9.69%	1.84%	1.32%
$r = 0.02$	0.77%	27.37%	8.98%	0.59%	0.38%
$r = 0.03$	0.42%	27.74%	8.74%	0.32%	0.31%
Direct output loss and Sanction					
$\lambda_s = 0.012$	2.16%	27.93%	9.69%	1.84%	1.32%
$\lambda_s = 0$	0.70%	26.30%	9.13%	0.66%	0.71%
$\lambda_d = 0.02$	2.16%	27.93%	9.69%	1.84%	1.32%
$\lambda_d = 0.03$	1.22%	23.54%	14.24%	1.08%	0.58%
$\lambda_d = 0.04$	0.69%	21.49%	18.96%	0.63%	0.51%

Table 7: Sensitivity Analysis for Benchmark Model

country has a higher bargaining power, the lower default frequency leads to less deadweight loss and smaller consumption volatility, therefore consumption increases.

Our results shed light on the impact of reform in sovereign bond restructuring on the international financial market.²³ Through the experiments on our model, we find that when the sovereign borrower has higher bargaining power, the country's borrowing cost does not necessarily increase. Yet the amount of sovereign debt issued on the market is greatly affected and the extent of risk sharing differs with the bargaining power. These results are consistent with the recent empirical findings on bond issuance and spreads in Eichengreen, Kletzer and Mody (2003).

5.3 Sensitivity Analysis

Table 7 presents the sensitivity analysis of our model to different parameter values.

The first panel shows the sensitivity of results to the choice of time discount factor. Because a more patient country cares more about its reentry to capital markets in the future, the value of renegotiation agreement is relatively higher than the cost of repaying more reduced debt. Therefore, the bargaining results in a higher recovery rate. The default

²³The use of Collective Action Clauses (CACs) is argued to be an improvement of the debt restructuring process in a recent debate. Because CACs can align bondholders' incentives by specifying a majority rule that binds all bondholders to eliminate the "hold out" problem. Thus CACs can be viewed as an experiment which assigns more bargaining power to the sovereign borrower. See Eichengreen, Kletzer and Mody (2003) and Weinschelbaum and Wynne (2005) for more details.

probability also decreases when the country is more patient because the intertemporal consumption smoothing is highly valued. Accordingly, the average bond spreads decrease with the discount factor.

The second panel reports the effect of risk free interest rates. A higher risk free rate implies higher borrowing costs for the country. Thus the country borrows less and defaults less frequently. At the same time, the average recovery rate changes slightly with the risk free rate. The total effect of lower default risk and a small change in recovery rate is that the bond spreads decrease with the risk free rate. This is consistent with what Eichengreen and Mody (1998) find in their empirical study of sovereign bond spreads.

Regarding the sanctions, the country implicitly has a higher value at its threat point in bargaining when the creditors do not have any sanction technology (i.e. $\lambda_s = 0$). Therefore, debt renegotiation results in a smaller recovery rate and the bond price schedule shifts down. In this case, the country's debt-to-output ratio and default probability are lower. Overall, average bond spreads decrease. Finally, an increase in the output loss in default (λ_d) lowers default probability because of the higher default penalty. But the debt recovery rate decreases because the creditors' sanction threat becomes relatively less important. As a result, the debt-to-output ratio increases. Average bond spreads decrease because the drop in default probability dominates.

6 Conclusion

It is well observed that sovereign debt crises have a great impact on the borrowing countries and international capital markets. Therefore, it is crucial to understand the sovereign default risk and the role of debt crises resolution in the sovereign debt markets. This paper studies sovereign default and debt renegotiation in a small open economy model. This model allows us to investigate the interaction between default and debt renegotiation within a dynamic borrowing framework. We find that debt recovery rates decrease with indebtedness and, in turn, affect the country's *ex ante* incentive to default. In equilibrium, sovereign bonds are priced to compensate creditors for the risks of default and restructuring. Consistent with the empirical evidence, the model predicts that interest rates increase with the level of debt.

We use the model to analyze quantitatively the sovereign debt of Argentina. The model successfully accounts for the high bond spreads, countercyclical country interest rates, and other key features of the Argentine economy. The model also replicates the dynamics of bond interest rates during the recent Argentine debt crisis. Furthermore, we demonstrate

that the changes in bargaining power have a great impact on debt recovery rates as well as on the sovereign bond spreads, shedding light on the policy implications of sovereign debt restructuring procedure. Overall, our study points out the importance of analyzing the connection between default and renegotiation in understanding sovereign debt market. One direction for future research is to investigate the role of international financial institutions in such a strategic interplay between default and renegotiation.

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Appendix

Proofs

Proof of Theorem 1. The proof consists of three steps.

Step 1. Given any debt recovery schedule $\alpha(b, y) \in A$, we define a price correspondence $\varphi(q)$ that takes points in Q .

$$\varphi(q)(b', y; \alpha) = \begin{cases} \begin{aligned} &(1 - p(q)(b', y; \alpha)) / (1 + r) \\ &+ p(q)(b', y; \alpha) \cdot \gamma(q)(b', y; \alpha) / (1 + r) \end{aligned} & \text{if } b' \geq 0 \\ 1 / (1 + r) & \text{if } b' \leq 0 \end{cases} \quad (17)$$

where $p(q)(b', y; \alpha)$ and $\gamma(q)(b', y; \alpha)$ satisfy (12) and (13). Thus, $\varphi(q)(b', y; \alpha)$ is the set of prices for a debt contract of type (b', y) that are consistent with zero profits given the price function q . We can show that $\varphi(q)(b', y; \alpha)$ is a closed interval in R and the correspondence $\varphi(q)(b', y; \alpha)$ has a closed graph (see Lemma App 5 and Lemma 8 in Chatterjee et al. (2002) for similar proofs). Therefore $\varphi(q)(b', y; \alpha)$ is an upper hemicontinuous correspondence. For any $q \in Q$, let $\varphi(q) \subset Q$ be the product correspondence $\prod_{b', y \in B \times Y} \varphi(q)(b', y; \alpha)$. Since $\varphi(q)(b', y; \alpha)$ is convex-valued for each b', y , $\varphi(q)$ is convex-valued as well. Furthermore, since $\varphi(q)(b', y; \alpha)$ is upper hemicontinuous with compact values for each b', y , the product correspondence $\varphi(q)$ is also upper hemicontinuous with compact values. (see Aliprantis and Border (1999), Thm 16.28). Therefore, $\varphi(q; \alpha)$ is a closed convex-valued correspondence that takes elements of the compact, convex set Q and returns sets in Q . By Kakutani-Fan-Glicksberg FPT (see Aliprantis and Border (1999), Thm 16.51) there is $q^* \in Q$ such that $q^* \in \varphi(q^*)$. Hence, there is an equilibrium bond price function $q^*(b', y)(\alpha)$ given the debt recovery schedule α .

Step 2. Given any bond price function $q(b, y) \in Q$, we define a debt recovery schedule correspondence $\psi(\alpha)$ that takes point in A .

$$\begin{aligned} \psi(\alpha)(b, y; q) &= \arg \max_{a \in [0, 1]} \left[(\Delta^B(a; b, y, q, \alpha))^\theta (\Delta^L(a; b, y, q, \alpha))^{1-\theta} \right] \\ &\text{s.t. } \Delta^B(a; b, y, q, \alpha) \geq 0 \\ &\quad \Delta^L(a; b, y, q, \alpha) \geq 0 \end{aligned} \quad (18)$$

$\psi(\alpha)(b, y; q)$ is the set of debt recovery rates for debt contract of type (b, y) that are consistent with Nash bargaining game.

Given q , for each b', y , $\psi(\alpha)(b', y; q)$ is an upper hemicontinuous correspondence with nonempty compact values from Berge's Maximum Theorem (see Aliprantis and Border (1999) Thm 16.31 and the technical appendix for details). For any $\alpha \in A$, let $\psi(\alpha; a) \subset A$ be the product correspondence $\prod_{b', y \in B \times Y} \psi(\alpha)(b', y; q)$. Since $\psi(q)(b', y; \alpha)$ is upper hemicontinuous with compact values for each b', y , the product correspondence $\psi(q; \alpha)$ is also upper hemicontinuous with compact values. (see Aliprantis and Border (1999), Thm 16.28). For bargaining power $\theta \in \Theta$, $\psi(\alpha)(b', y; q)$ is single-valued, so is the product correspondence $\psi(q; \alpha)$. Therefore, $\psi(q; \alpha)$ is a closed convex-valued correspondence that takes elements of the compact, convex set A and returns sets in A . By Kakutani-Fan-Glicksberg FPT (see Aliprantis and Border (1999), Thm 16.51) there is $\alpha^* \in A$ such that $\alpha^* \in \omega(\alpha^*; q)$. Hence, there exists an equilibrium debt recovery schedule $\alpha^*(b', y)(\alpha)$ given the bond price function q .

Step 3. We construct a functional mapping operator $T : Q \times A \rightarrow Q \times A$ such that

$$T(q, \alpha)(b, y) = \begin{bmatrix} \varphi(q)(b, y; q, \alpha) \\ \psi(\alpha)(b, y; q, \alpha) \end{bmatrix}$$

Because $\varphi(q)(b', y; q, \alpha)$ and $\psi(\alpha)(b, y; q, \alpha)$ are upper hemicontinuous, $T(q, \alpha)$ is upper hemicontinuous. (see Aliprantis and Border (1999) Thm 16.23). Therefore, the correspondence $T(q, \alpha)$ has a closed graph. We can also show that $T(q, \alpha)$ is convex valued. Suppose $(q_1, \alpha_1) \in T(q, \alpha)$ and $(q_2, \alpha_2) \in T(q, \alpha)$. Because $\varphi(q)(b', y; q, \alpha)$ is convex valued, $\gamma q_1 + (1 - \gamma) q_2 \in \varphi(q; \alpha)$. Because $\psi(\alpha)(b, y; q, \alpha)$ is single valued, $\alpha_1 = \alpha_2 = \gamma \alpha_1 + (1 - \gamma) \alpha_2 \in \psi(\alpha; q)$. Therefore, $(\gamma q_1 + (1 - \gamma) q_2, \gamma \alpha_1 + (1 - \gamma) \alpha_2) \in T(q, \alpha)$. Hence, we can apply Kakutani's fixed point theorem and show the existence of a fixed point.

$$T(q^*, \alpha)(b, y) = (q^*, \alpha^*)$$

A recursive equilibrium exists. ■

Proof of Theorem 2. Because $\Delta^B(a; b, y)$ and $\Delta^L(a; b, y)$ are both function of ab , define $\Delta^B(a; b, y) = \tilde{\Delta}^B(ab; y)$, and $\Delta^L(a; b, y) = \tilde{\Delta}^L(ab; y)$. The bargaining problem is equivalent to the following

$$\begin{aligned} \max_{ab} & \left[\left(\tilde{\Delta}^B(ab; y) \right)^\theta \left(\tilde{\Delta}^L(ab; y) \right)^{1-\theta} \right] \\ \text{s.t.} & \tilde{\Delta}^B(ab; y) \geq 0 \\ & \tilde{\Delta}^L(ab; y) \geq 0 \end{aligned}$$

where the functional form of $\tilde{\Delta}^B(ab; y)$ and $\tilde{\Delta}^L(ab; y)$ are transformations of $\Delta^B(a; b, y)$ and $\Delta^L(a; b, y)$. For bargaining power $\theta \in \Theta$, given (b, y) , the renegotiation surplus has a unique optimum. In the transformed problem, the optimal solution is solely a function of endowment y and we denote it as $b_y \leq 0$. The bargaining over debt reduction has constraint

$a \in [0, 1]$. When $b \leq b_y$, the constraint $a \in [0, 1]$ is not binding, so $a = \frac{b_y}{b}$. If $b \geq b_y$, the constraint $a \in [0, 1]$ is binding, so $a = 1$.

Therefore,

$$\psi(\alpha; q)(b, y) = \begin{cases} \frac{b_y}{b} & \text{if } b \leq b_y \\ 1 & \text{if } b \geq b_y \end{cases}$$

Because an equilibrium debt recovery rate function is a fixed point of the correspondence $\psi(\alpha; q)(b, y)$, the debt recovery rate also satisfies

$$\alpha(b, y) = \begin{cases} \frac{b_y}{b} & \text{if } b \leq b_y \\ 1 & \text{if } b \geq b_y \end{cases}$$

■

Proof of Theorem 3. Since the equilibrium debt recovery schedule satisfies Theorem 2, given endowment y , the debt arrear after defaulting is independent of b . Thus, the utility from defaulting is independent of b . We can also show that the utility from not defaulting $v(b, 0, y)$ is increasing in b . (The proof follows Lemma 2 in Chatterjee et al 2002.) Therefore, if $v(b^1, 0, y) = u((1 - \lambda_d)y) + \beta v(b_y, 1, y)$, then it must be the case that $v(b^0, 0, y) = u((1 - \lambda_d)y) + \beta v(b_y, 1, y)$. Hence, any y that belongs in $\overline{D}(b^1)$ must also belong in $\overline{D}(b^0)$. ■

Proof of Theorem 4. Let $d^*(b, 0, y')$ be the equilibrium default functions. Equilibrium default probability is then given by

$$p(b', y) = \int_Y d^*(b', 0, y') d\mu(y'|y)$$

From Theorem 3, if $d^*(b^1, 0, y') = 1$, then $d^*(b^0, 0, y') = 1$. Therefore,

$$p(b^0, y) \geq p(b^1, y)$$

■

Proof of Theorem 5. Let $p^*(b, y)$ be the equilibrium default probability function and $\alpha^*(b, y)$ be the equilibrium debt recovery schedule. The expected debt recovery rate is then given by

$$\gamma(b', y) = \frac{\int_Y d(b', 0, y') \alpha(b', y') d\mu(y'|y)}{\int_Y d(b', 0, y') d\mu(y'|y)}$$

From Theorem 2, given y , for $b^0 < b^1 \leq b_y \leq 0$, $\alpha^*(b^0, y) < \alpha^*(b^1, y) \leq 1$. Therefore, the equilibrium expected debt recovery rate $\gamma^*(b^0, y) < \gamma^*(b^1, y) \leq 1$. And from Theorem 4, $p^*(b^0, y) \geq p^*(b^1, y)$. For the country's indebtedness, the equilibrium bond price is given by

$$\begin{aligned} q(b', y) &= \frac{1 - p(b', y)}{1 + r} + \frac{p(b', y) \cdot \gamma(b', y)}{1 + r} \\ &= \frac{1 - p(b', y)(1 - \gamma(b', y))}{1 + r} \end{aligned}$$

Hence, we obtain that

$$q(b^0, y) \leq q(b^1, y)$$

■

Proof of Theorem 6. Because $u(\cdot)$ is concave function, given b , for all $b' \leq 0$,

$$\frac{d}{db'} [u((1 - \lambda_d)y + b) - u((1 - \lambda_d)y + b - b'/(1 + r))] \geq 0$$

If $b \in B_{--}$ and $b \geq b$, for all $b' < 0$,

$$\begin{aligned} & u((1 - \lambda_d)y + b) - u((1 - \lambda_d)y + b - b'/(1 + r)) \\ & \geq u((1 - \lambda_d)y + b) + u((1 - \lambda_d)y + b - b'/(1 + r)) \\ & \geq \beta y^{1-\sigma} \int_Y v(0, 0, y') d\mu(y'|y) + \beta \int_Y v(b', 1, y') d\mu(y'|y) \end{aligned}$$

Thus,

$$\begin{aligned} & u((1 - \lambda_d)y + b) + \beta \int_Y v(0, 0, y') d\mu(y'|y) \\ & \geq \sup_{b' < 0} u((1 - \lambda_d)y + b - b'/(1 + r)) + \beta \int_Y v(b', 1, y') d\mu(y'|y) \end{aligned}$$

which implies

$$v(b, 1, y) = u((1 - \lambda_d)y + b) + \beta \int_Y v(0, 0, y') d\mu(y'|y)$$

If $b \in B_{--}$ and $b \geq b$, for all $b' < 0$, suppose

$$\begin{aligned} & u((1 - \lambda_d)y + b) + \beta \int_Y v(0, 0, y') d\mu(y'|y) \\ & > \sup_{b' < 0} u((1 - \lambda_d)y + b - b'/(1 + r)) + \beta \int_Y v(b', 1, y') d\mu(y'|y) \end{aligned}$$

then, according to the above analysis,

$$\begin{aligned} & u((1 - \lambda_d)y + b) + \beta \int_Y v(0, 0, y') d\mu(y'|y) \\ & > \sup_{b' < 0} u((1 - \lambda_d)y + b - b'/(1 + r)) + \beta \int_Y v(b', 1, y') d\mu(y'|y) \end{aligned}$$

contradiction. ■

Computation Algorithm

The procedure to compute the equilibrium of the model economy is the following:

First we set grids on the spaces of asset holdings and endowment. The asset space and the space for endowment shocks are discretized into fine grids. The limits of the asset space are set to ensure that the limits do not bind in equilibrium. The limits of endowment space are large to include big deviations from the average value of shocks. We approximate the distribution of endowment shock using a discrete Markov transition matrix. Then, we use

the following procedure to compute an equilibrium.

1. Guess an initial debt recovery schedule $\alpha^{(0)}$.

2. Given a debt recovery schedule $\alpha^{(0)}$, we solve for equilibrium bond price $q^{(0)}$

(a) Guess an initial price of discounted loans $q^{(00)}$.

(b) Given a price for loans, $q^{(00)}$, we solve the country's optimization problem. This procedure includes finding the value function as well as the default decisions. We first guess value function $v^{(0)}$ and iterate it using the Bellman equation to find the fixed out v^* , given bond price and debt recovery rates. For the problem of a country with debt and a good credit score, we also obtain the optimal default choice, which requires comparison between the implications of defaulting and not defaulting. This comparison also enables us to calculate the corresponding default set.

(c) Using the default set derived in step (b) and the zero profit condition for international investors, we compute the new price of discounted bonds $q^{(01)}$. If $q^{(01)}$ is sufficiently close to $q^{(00)}$, stop iterating on q , assign $q^{(01)}$ to $q^{(0)}$ and go on to the step 3, otherwise go back to step (b).

3. Solve the bargaining problem given converged bond price $q^{(0)}$ and compute the new debt recovery schedule for every (b, y) . If the new debt recovery schedule $\alpha^{(1)}$ is sufficiently close to $\alpha^{(0)}$, stop iterating on α , otherwise, go back to step 2.