BARRO and SALA-I-MARTIN: JOEG, 1997

Production

\[
Y_i = A_i (L_i)^{1-\alpha} \sum_{j=1}^{N_i} (X_{ij})^\alpha \quad i = 1, 2
\]

Output = Expenditures

\[
Y_i = C_i + \sum_{j=1}^{N_i} X_{ij} + RD
\]

\(C_i\) and \(X_{ij}\) require one unit of \(Y_i\). Invention of new variety requires \(\eta_i\). Assume \(N_1(0) > N_2(0)\), and new discoveries occur in country 1.

Country 1:

Monopoly profit flow to inventor of intermediate good \(j\):

\[
\pi_{1j} = (P_{1j} - 1) X_{1j}
\]

From Prod. fn:

\[
\frac{\partial Y_1}{\partial X_{1j}} = A_1 \alpha (L_1)^{1-\alpha} (X_{1j})^{\alpha-1}
\]

If we set this marginal product to the price of the intermediate good, \(P_{1j}\), and setting the price of \(Y_1 = 1\):

\[
X_{1j} = L_1 \left( \frac{A_1 \alpha}{P_{1j}} \right)^{\frac{1}{\alpha+1}}
\]

Intermediate good profit function then:

\[
\text{Max}_{P_{1j}} \left( P_{1j} - 1 \right) L_1 \left( \frac{A_1 \alpha}{P_{1j}} \right)^{\frac{1}{\alpha+1}}
\]

\[
\text{Max}_{P_{1j}} L_1 \left( A_1 \alpha \right) \frac{1}{\alpha+1} \left( \frac{1}{1-\alpha} \right) \left( P_{1j} \right)^{\frac{1}{\alpha+1}} \left( P_{1j} \right)^{\frac{2-\alpha}{\alpha+1}}
\]

\[
0 = \left( \frac{1}{1-\alpha} \right) \left( P_{1j} \right)^{\frac{1}{\alpha+1}} \left( \alpha - P_{1j} \right)
\]

\[
P_{1j} = P_1 = \alpha^{-1}
\]

Substitution into \(X_{1j}\) and then \(Y_1\):

\[
X_{1j} = X_1 = L_1 (A_1)^{\frac{1}{\alpha+1}} (\alpha)^{\frac{2}{\alpha+1}}
\]

\[
Y_1 = (A_1)^{\frac{1}{\alpha+1}} \alpha^{\frac{2\alpha}{\alpha+1}} L_1 N_1
\]

\[
\frac{Y_1}{L_1} = (A_1)^{\frac{1}{\alpha+1}} \alpha^{\frac{2\alpha}{\alpha+1}} N_1
\]
Output per worker rises with $A_1$, $N_1$.

Monopoly profit flow:

$$\pi_{1j} = \pi_1 = (1 - \alpha)L_1 (A_1)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$$

with present value:

$$V_1(t) = \pi_1 \int_t^\infty e^{-\int_s^t r_1(\nu) d\nu} ds$$

If there is free entry, $V_1(t) = \eta_1$, which implies $r_1$ is constant:

$$r_1 = \frac{\pi_1}{\eta_1}$$

So rate of return is the ratio of profit flow over cost.

Consumers:

$$U = \int_0^\infty e^{-\rho t} (1 - \theta)^{-1} (e^{1-\theta} - 1) dt$$

FOC:

$$\frac{\dot{C}_1}{C_1} = \theta^{-1} (r_1 - \rho)$$

If $C_1$, $N_1$ and $Y_1$ grow at the same rate, assuming $\frac{\pi_1}{\eta_1} - \rho > 0$ (since $N_1$ cannot decrease),

$$\gamma = \frac{\dot{N}}{N} = \theta^{-1} (r_1 - \rho) = \theta^{-1} \left( \frac{\pi_1}{\eta_1} - \rho \right)$$

$$= \theta^{-1} \left( \left( \frac{1-\alpha}{\alpha} \right) (\eta_1)^{-1} L_1 (A_1)^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} - \rho \right)$$

To show that $C_1$, and $Y_1$ grow at the same rate as $N_1$:

$$Y_1 = (A_1)^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L_1 N_1$$

so $Y_1$ grows as $N_1$.

$$C_1 = Y_1 - \eta N_1 - N_1 X_1 = Y_1 - \eta \gamma N_1 - N_1 X_1$$

$$= (A_1)^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} L_1 N_1 - \eta \gamma N_1 - N_1 L_1 (A_1)^{\frac{1}{1-\alpha}} (\alpha)^{\frac{\alpha}{1-\alpha}}$$

so $C_1$ grows as $N_1$.

COUNTRY 2

Adapting intermediate goods cost $V_2(t) = v_2 \left( \frac{N_2}{N_1} \right) < \eta_2$ for $\frac{N_2}{N_1} < 1$, and $n_2$ otherwise. Results are parallel to country 1:

$$P_{2,j} = \alpha^{-1}$$
\[ X_{2j} = X_2 = L_2 (A_2)^{\frac{1-\alpha}{\alpha}} (\alpha)^{\frac{1-\alpha}{\alpha}} \]
\[ Y_2 = (A_2)^{\frac{2-\alpha}{\alpha}} L_2 N_2 \]
\[ \frac{Y_2}{L_2} = (A_2)^{\frac{2-\alpha}{\alpha}} N_2 \]
\[ \pi_{2j} = \pi_2 = (1-\alpha) L_2 (A_2)^{\frac{1+\alpha}{\alpha}} \]

So
\[ \frac{Y_2}{Y_1} = \left(\frac{A_2}{A_1}\right)^{\frac{1-\alpha}{\alpha}} \left(\frac{N_2}{N_1}\right) \]

Also,
\[ v_2 \left(\frac{N_2}{N_1}\right) = \pi_2 \int_t^\infty e^{-\int_t^u r_2(v)dv} ds \]

implies
\[ r_2 = \frac{\pi_2}{v_2} + \frac{\dot{v}_2}{v_2} \]

because, differentiating \[ v_2 \left(\frac{N_2}{N_1}\right) = \pi_2 \int_t^\infty e^{-\int_t^u r_2(v)dv} ds \],

\[ \dot{v}_2(t) = \pi_2(-1 + r_2(t)) \int_t^\infty e^{-\int_t^u r_2(v)dv} ds \]
\[ = -\pi_2 + r_2(t) v_2(t) \]
\[ v_2(t) = \frac{\pi_2 + \dot{v}_2(t)}{r_2(t)} \]
\[ \frac{\dot{v}_2(t)}{v_2(t)} + \pi_2 = r_2(t) \]

Note that here \( r_2 \) is not simply \( \frac{\pi_2}{v_2} \), but includes the capital gain term \( \frac{\dot{v}_2}{v_2} \), as it reflects the changing cost of entry that is reflected in the monopoly profits.

Also:
\[ \frac{\dot{C}_2}{C_2} = \theta^{-1} (r_2 - \rho) \]

STEADY STATE GROWTH

\[ \frac{\dot{N}_i}{N_i} = \frac{\dot{C}_i}{C_i} = \frac{\dot{Y}_i}{Y_i} = \gamma_1 = \gamma_2 = \gamma \]

\[ \left(\frac{N_2}{N_1}\right)^* = \text{const} \]
\[ r_1 = r_2 = \frac{\pi_1}{\eta_1} = \frac{\pi_2}{v_2} \]
\[ v_2 = \eta_1 \left(\frac{\pi_2}{\pi_1}\right) = \eta_1 \left(\frac{A_2}{A_1}\right)^{\frac{1+\alpha}{\alpha}} \left(\frac{L_2}{L_1}\right) \]

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But, \( v_2 < \eta_2 \), so if
\[
\eta_1 \left( \frac{A_2}{A_1} \right)^{\frac{1}{1-\alpha}} \left( \frac{L_2}{L_1} \right) < \eta_2
\]
then country 1 is always the leader with no incentive to imitate since \( N_1 > N_2 \), and country 2 always imitates and does not innovate since \( v_2 < \eta_2 \).

**DYNAMICS**

Now let
\[
v_2 = \eta_2 \left( \frac{N_2}{N_1} \right)^{\sigma}
\]

Then
\[
\left( \frac{N_2}{N_1} \right)^* = \left( \frac{\eta_1 A_2}{\eta_2 A_1} \right)^{\frac{1}{1-\alpha}} \left( \frac{L_2}{L_1} \right)^{\frac{1}{\sigma}}
\]

Let \( H = \left( \frac{N_2}{N_1} \right) \)

Using \( r_2 = \frac{\pi_2}{v_2} + \frac{\dot{v}_2}{v_2} \),

\[
\begin{align*}
\frac{\dot{C}_2}{C_2} &= \theta^{-1} (r_2 - \rho) = \theta^{-1} \left( \frac{\pi_2}{v_2} + \frac{\dot{v}_2}{v_2} - \rho \right) \\
\frac{\dot{C}_2}{C_2} &= \theta^{-1} \left( \frac{\pi_2}{v_2} + \sigma \frac{\dot{H}}{H} - \rho \right)
\end{align*}
\]

Now
\[
\begin{align*}
v_2 \dot{N}_2 &= Y_2 - C_2 - N_2 X_2 \\
\dot{N}_2 &= \left( \frac{1}{v_2} \right) (Y_2 - C_2 - N_2 X_2) \\
\frac{\dot{N}_2}{N_2} &= \left( \frac{1}{v_2} \right) \left( Y_2 - C_2 - N_2 X_2 \right) \\
\frac{\dot{N}_2}{N_2} &= \left( \frac{1}{v_2} \right) \left( L_2 (A_2)^{\frac{1}{1-\alpha}} \alpha^{\frac{2-\alpha}{1-\alpha}} - \frac{C_2}{N_2} \right) \\
\frac{\dot{N}_2}{N_2} &= \left( \frac{1}{v_2} \right) \left( L_2 (A_2)^{\frac{1}{1-\alpha}} \alpha^{\frac{2-\alpha}{1-\alpha}} (\alpha - 1) - \frac{C_2}{N_2} \right) \\
\frac{\dot{N}_2}{N_2} &= \left( \frac{1}{v_2} \right) \left( L_2 (A_2)^{\frac{1}{1-\alpha}} \alpha^{\frac{2-\alpha}{1-\alpha}} \left( \frac{1}{\alpha} - 1 \right) - \frac{C_2}{N_2} \right)
\end{align*}
\]

**BUT**

\[
\begin{align*}
\pi_2 &= (1 - \alpha) L_2 (A_2)^{\frac{1}{1-\alpha}} \alpha^{\frac{2-\alpha}{1-\alpha}} \\
\frac{(1 + \alpha)}{\alpha} \pi_2 &= \frac{(1 + \alpha)}{\alpha} \frac{(1 - \alpha)}{\alpha} L_2 (A_2)^{\frac{1}{1-\alpha}} \alpha^{\frac{2-\alpha}{1-\alpha}}
\end{align*}
\]
SO

\[
\frac{\dot{N}_2}{N_2} = \left( \frac{1}{v_2} \right) \left( L_2(A_2) \frac{\alpha}{1-\alpha} \frac{\alpha}{\alpha} \frac{\alpha}{\alpha} \right)
\]

\[
\frac{\dot{N}_1}{N_2} = \left( \frac{1}{v_2} \right) \left( \frac{(1+\alpha)}{\alpha} \pi_2 - C_2 \right)
\]

Let \( \chi_2 = \frac{C_2}{\pi_2} \). Then

\[
\frac{\dot{H}}{H} = \frac{\dot{N}_2}{N_2} - \frac{\dot{N}_1}{N_1} = \left( \frac{1}{v_2} \right) \left( \frac{(1+\alpha)}{\alpha} \pi_2 - \chi_2 \right) - \gamma_1
\]

Now

\[
\frac{\dot{C}_2}{C_2} = \theta^{-1} \left( \frac{\pi_2}{v_2} + \sigma \frac{\dot{H}}{H} - \rho \right)
\]

\[
= \theta^{-1} \left( \frac{\pi_2}{v_2} + \sigma \left( \frac{1}{v_2} \left( \frac{(1+\alpha)}{\alpha} \pi_2 - \chi_2 \right) - \gamma_1 \right) \right)
\]

\[
= \theta^{-1} \left( \frac{1}{v_2} \right) \left( \pi_2 + \sigma \left( \frac{(1+\alpha)}{\alpha} \pi_2 - \chi_2 \right) - \gamma_1 \right)
\]

However

\[
\frac{\dot{\chi}_2}{\chi_2} = \frac{\dot{C}_2}{C_2} - \frac{\dot{N}_2}{N_2}
\]

\[
\frac{\dot{\chi}_2}{\chi_2} = \left( \frac{1}{v_2} \right) \left( \pi_2 + \sigma \left( \frac{(1+\alpha)}{\alpha} \pi_2 - \chi_2 \right) \right) - \theta^{-1} \left( \rho + \sigma \gamma_1 \right)
\]

So the two dynamic equations are:

\[
\frac{\dot{H}}{H} = \left( \frac{1}{v_2(H)} \right) \left( \frac{(1+\alpha)}{\alpha} \pi_2 - \chi_2 \right) - \gamma_1
\]

\[
\frac{\dot{\chi}_2}{\chi_2} = \left( \frac{1}{\theta v_2(H)} \right) \left( \pi_2 + \sigma \left( \chi_2 - \frac{(1+\alpha)}{\alpha} \pi_2 \right) \right) - \theta^{-1} \left( \rho + \sigma \gamma_1 \right)
\]

where \( v_2 \) depends on \( H = \frac{N_2}{N_1} \) since we had assumed \( v_2 = \eta_2 \left( \frac{N_2}{N_1} \right)^{\gamma} \).

Upward sloping saddle path (for the two cases where \( \theta > \sigma \) and also for \( \theta < \sigma \)) implies that \( N_2 \) grows faster than \( N_1 \) in approaching the steady state ratio \( \left( \frac{N_2}{N_1} \right)^{\gamma} \). Since \( Y_i \) is proportional to \( N_i \), \( Y_2 \) grows faster than \( Y_1 \). Eventually at steady state both countries grow at \( \gamma_1 \).

Evaluate Jacobian:

\[
J = \begin{pmatrix}
-v_2^{-2} v_2' \left( \frac{(1+\alpha)}{\alpha} \pi_2 - \chi_2 \right) H & -v_2^{-1} H \\
-v_2^{-2} \theta^{-1} v_2' \left( \pi_2 + \left( \theta - \sigma \right) \left( \chi_2 - \frac{(1+\alpha)}{\alpha} \pi_2 \right) \right) \chi_2 & v_2^{-1} \theta^{-1} \left( \theta - \sigma \right) \chi_2
\end{pmatrix}
\]

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DET

\[-\chi_2 H v_2^{-2} v_2' \left( \frac{1 + \alpha}{\alpha} \pi_2 - \chi_2 \right) v_2^{-1} \theta^{-1} (\theta - \sigma) + v_2' \chi_2 H v_2^{-2} \theta^{-1} \left( \pi_2 + (\theta - \sigma) \left( \chi_2 - \frac{(1 + \alpha)}{\alpha} \pi_2 \right) \right) v_2^{-1} \]

\[= -\chi_2 H v_2^{-2} \theta^{-1} v_2' \left( \frac{1 + \alpha}{\alpha} \pi_2 - \chi_2 \right) (\theta - \sigma) + \left( \pi_2 + (\theta - \sigma) \left( \chi_2 - \frac{(1 + \alpha)}{\alpha} \pi_2 \right) \right) \]

\[= -\chi_2 H v_2^{-3} \theta^{-1} v_2' \pi_2 < 0 \rightarrow \text{Saddle Path} \]

TRACE

\[-v_2^{-2} \left( \frac{1 + \alpha}{\alpha} \pi_2 - \chi_2 \right) H + v_2^{-1} \theta^{-1} (\theta - \sigma) \chi_2 \]

Steady State

\[\left( \pi_2 + (\theta - \sigma) \left( \chi_2 - \frac{(1 + \alpha)}{\alpha} \pi_2 \right) \right) = \theta^{-1} (\rho + \sigma \gamma_1) \theta v_2 (H) \]

\[\frac{(\pi_2 - (\theta - \sigma) (\gamma_1 v_2 (H)))}{\pi_2} = \theta^{-1} (\rho + \sigma \gamma_1) \theta v_2 (H) \]

\[\frac{(\theta - \sigma) \gamma_1 + (\rho + \sigma \gamma_1)}{\pi_2} = v_2 (H) \]

\[\frac{\pi_2}{(\theta \gamma_1 + \rho)} = v_2 (H) \]

\[\pi_2 \left( \frac{1 + \alpha}{\alpha} - \frac{\gamma_1}{(\theta \gamma_1 + \rho)} \right) = \chi_2 \]

ALTERNATE DIFFUSION (Careful with notation on $H = \frac{N_i}{N_t}$, but $H_i$ refers to human capital of country $i$)

We will examine the implications of two types processes often studied in the context of disaggregated models of technology diffusion [Banks (1994)]. We can express the original Nelson-Phelps model of technology diffusion as follows:

\[
\frac{\dot{A}_i(t)}{A_i(t)} = g_i(H_i(t)) + c(H_i(t)) \left( \frac{A_m(t)}{A_i(t)} - 1 \right) \tag{1}
\]

where $A_i(t)$ is the TFP, $g_i(H_i(t))$ is the component of TFP growth that depends on the level of education $H_i(t)$ in country $i$ and $c(H_i(t)) \left( \frac{A_m(t)}{A_i(t)} - 1 \right)$ represents the rate of technology diffusion from the leader country $m$ to country $i$. We assume that $c_i(\cdot)$ and $g_i(\cdot)$ are increasing functions. The level of education $H_i(t)$ affects the rate at which the technology gap $\left( \frac{A_m(t)}{A_i(t)} - 1 \right)$ is closed. If the ranking of $g_i(H_i(t))$ across countries do not change, or if $H_i$ is constant, a technology leader will emerge in finite time with $g_m = g(H_m(t)) > g(H_i(t)) = g_i$. After that the leader will grow at rate $g_m$ and the followers will fall behind in levels of TFP until the point at which their growth rate will match the leader’s
growth rate $g_m$. This can be seen from the solution of the above equation when $H_i$'s are constant:

$$A_i(t) = (A_i(0) - \Omega A_m(0)) e^{(g_i - c_i)t} + \Omega A_m(0) e^{g_m t}$$

(2)

where $c_i = c(H_i)$, $g_i = g(H_i)$ and

$$\Omega = \frac{c_i}{g_i - c_i + g_m} > 0.$$ 

It is clear, since $g_m > g_i$, that

$$\lim_{t \to \infty} \frac{A_i(t)}{A_m(t)} = \Omega$$

This is, for all parametrizations, a world balanced growth path with the leader acting as the “locomotive.” Technology diffusion and “catch-up” assures that despite scale effects and educational differences, all countries eventually grow at the same rate.\(^2\)

The technology diffusion and catch-up processes outlined above are also known as the confined exponential or Coleman diffusion process (see Banks(1994) and Sharif and Ramanthran (1981)) An alternative formulation that is similar in spirit is the logistic or Dodd model of technology diffusion (see Sharif and Ramanthran (1981)). It is given by

$$\frac{\dot{A}_i(t)}{A_i(t)} = g(H_i(t)) + c(H_i(t)) \left(1 - \frac{A_i(t)}{A_m(t)}\right)$$

(3)

The difference between the dynamics under the logistic model of technology diffusion and the confined exponential one is due to the presence of the extra term $\left(\frac{A_i(t)}{A_m(t)}\right)$. This term acts to dampen the rate of diffusion as the distance to the leader increases, reflecting perhaps the difficulty of adopting distant technologies as emphasized by Basu and Weil (1998). Catch-up therefore is slower when the the leader is either too large or too small. and is is fastest at intermediate distances.

\(^{1}\)The general solution when $H_i$'s are not constant is given by:

$$A_i(t) = A_i(0) e^{-\int_0^t (g(H_i(s)) - c(H_i(s))) ds}$$

$$\cdot \left[1 + \frac{1}{A_i(0)} \left( \int_0^t c(H_i(\tau)) (A_m(0) e^{\int_0^\tau g(H_m(\zeta)) d\zeta} e^{\int_0^\tau (g(H_i(\xi)) - c(H_i(\xi))) d\xi d\tau} \right) \right]$$

\(^{2}\)Note however that in transition, the higher is initial $A_i(0)$, the smaller is the technology gap to the leader and therefore the slower is the growth. This negative dependence on initial conditions is similar to standard convergence results in the neoclassical growth model, but the logic of catch-up is different.
If we assume, as before, that \( H_i \)'s (and therefore, \( c_i \)'s and \( g_i \)'s) are constant such that \( H_m > H_i \), and therefore that \( c(H_m) > c(H_i) \), then the solution to the logistic technology diffusion equation is given by

\[
A_i (t) = \frac{A_i (0) e^{(g_i + c_i)t}}{1 + \frac{A_i (0)}{A_m (0)} \left( \frac{c_i}{c_i + g_i - g_m} \right) \left( e^{(c_i + g_i - g_m)t} - 1 \right)} > 0
\]

This equation can be written as

\[
A_i (t) = \frac{A_m (0) e^{g_m t}}{e^{-(c_i + g_i - g_m)t} \left( \frac{A_m (0)}{A_i (0)} \right) + \frac{c_i}{(c_i + g_i - g_m)^2}}
\]

so that in the limit,

\[
\lim_{t \to \infty} A_i (t) = \begin{cases} 
\frac{c_i + g_i - g_m}{A_m (0)} & \text{if } (c_i + g_i - g_m) > 0 \\
0 & \text{if } (c_i + g_i - g_m) = 0 \\
\frac{c_i}{(c_i + g_i - g_m)^2} & \text{if } (c_i + g_i - g_m) < 0
\end{cases}
\]

BACK TO BARRO

\[
\dot{N}_2 = \left( \frac{1}{v_2} \right) \left( Y_2 - C_2 - N_2 X_2 \right)
\]

For simplicity assume \( C_2 = \mu Y_2 \)

\[
\dot{N}_2 = \left( \frac{1}{v_2} \right) \left( Y_2 (1 - \mu) - N_2 X_2 \right)
\]

\[
\frac{\dot{H}}{H} = \frac{\dot{N}_2}{N_2} - \frac{\dot{N}_1}{N_1} = \left( \frac{1}{v_2} \right) \left( \frac{Y_2}{N_2} - X_2 \right) - \gamma_1
\]

\[
\frac{\dot{H}}{H} = \frac{\dot{N}_2}{N_2} - \frac{\dot{N}_1}{N_1} = \left( \frac{1}{v_2} \right) \left( \frac{(1 + \alpha)}{\alpha} \pi_2 \right) - \gamma_1
\]

LET \( v_2 = (1 - H)^{-1} \rightarrow \text{Logistic} \)

LET \( v_2 = (H^{-1} - 1)^{-1} \rightarrow \text{exponential} \)

---

3Provided that \( (c_i + g_i - g_m) \neq 0 \). If \( (c_i + g_i - g_m) = 0 \), then the equation reduces to exponential form \( A_i (t) = A_i (0) e^{(g_i + c_i)t} \).

4The general solution where \( H_i \)'s are functions of time can be computed by defining \( B_i = (A_i)^{-1} \) and and transforming the logistic form into the confined exponential. After some computations, the general form can be obtained as

\[
A_i (t) = \frac{A_i (0) e^{\int_0^t \left( g(H_i(t)) + c(H_i(t)) \right) ds}}{1 + A_i (0) \left( \int_0^t c(H_i (\tau)) \left( A_m (0)^{-1} \right) \left( e^{\int_0^\tau g(H_m (\zeta)) d\zeta} \right) e^{\int_0^\tau \left( g(H_i (\xi)) + c(H_i (\xi)) \right) d\xi, d\tau \right) \left( e^{\int_0^\tau g(H_i (\xi)) d\xi} - 1 \right)}
\]

5\( A_i (t) > 0 \) because when \( c_i + g_i - g_m \neq 0 \), \( \frac{e^{(c_i + g_i - g_m)t}}{e^{(c_i + g_i - g_m)t} - 1} > 0 \).