Solow Model

\[ \dot{K} = sF(K, L) - \delta K \]
\[ = sLF(k, 1) - \delta K \]
\[ \frac{\dot{K}}{L} = sf(k) - \delta k \]
\[ \dot{k} = \frac{d\left(\frac{K}{L}\right)}{dt} = \frac{\dot{K}}{L} - nk \]

where \( \frac{\dot{L}}{L} = n \)
\[ \dot{k} = sf(k) - (\delta + n)k \]

Golden Rule
At steady state \( \dot{k} = 0 \), and

and

\[ \max_k c = f(k) - (\delta + n)k \]
\[ f'(k) = (\delta + n) \]
\( \beta - \text{Convergence (not conditional)}: \)

\[
\gamma = \frac{\dot{k}}{k} = s \frac{f(k)}{k} - (\delta + n)
\]

\[
\frac{d\gamma}{dk} = s \left( \frac{kf'(k) - f(k)}{k^2} \right) = \frac{s}{k} \left( f'(k) - \frac{f(k)}{k} \right)
\]

\[= \frac{s}{k} (MPK - APK) < 0\]

**Technical Progress:**

\[
\dot{K} = sF(K, A(t)L) - \delta K
\]

\[
\frac{\dot{L}}{L} = n; \quad \frac{\dot{A}}{A} = \theta; \quad x(t) = \frac{K(t)}{A(t)L(t)}
\]

\[
\dot{x} = sf(x) - (\delta + n + \theta)x
\]

\[
\gamma = \frac{\dot{x}}{x} = s \frac{f(x)}{x} - (\delta + n + \theta)
\]
The Solow Residual

Production Function (CRS):

\[ Q = AF(K,L) \]

\[ \dot{Q} = F_k \dot{K} + F_L \dot{L} + \dot{A}F \]

\[ \frac{\dot{Q}}{Q} = \left( \frac{KF_k}{Q} \right) \frac{\dot{K}}{K} + \left( \frac{LF_L}{Q} \right) \frac{\dot{L}}{L} + \frac{\dot{A}}{A} \]

\[ \frac{\dot{A}}{A} = \frac{\dot{Q}}{Q} - \left( \frac{KF_k}{Q} \right) \frac{\dot{K}}{K} - \left( \frac{LF_L}{Q} \right) \frac{\dot{L}}{L} \]

Let

\[ k = \frac{K}{L}, \quad q = \frac{Q}{L} = Af(k) \]

\[ \dot{q} = \dot{A}f + Af' \dot{k} \]

\[ \frac{\dot{q}}{q} = \frac{\dot{A}}{A} + \left( \frac{Af'(k)}{Q} \right) \frac{\dot{k}}{k} = \frac{\dot{A}}{A} + w_k \frac{\dot{k}}{k} \]

Solow adjusted capital for utilization (multiplied by \( \frac{employment}{Labor\ force} \)), used non-farm output for \( q \), used capital share of 33% from the data, and set \( A_{1909} = 1 \).
Then
\[
\frac{\Delta q}{q} = \frac{\Delta A}{A} + w_k \frac{\Delta k}{k}
\]
88% of growth accounted by \(\frac{\Delta A}{A}\), 12% by \(\frac{\Delta k}{k}\).

If there is a random element to productivity, \(\frac{\dot{A}}{A} = \mu + \theta_t\)

\[
SR_t = \frac{\dot{A}}{A} = \mu + \theta_t = \frac{\dot{Q}}{Q} - \left( \frac{KF_k}{Q} \right) \frac{\dot{K}}{K} - \left( \frac{LF_L}{Q} \right) \frac{\dot{L}}{L}
\]

Issues: (Denison).

Now, if there are variables in the economy that affect output or employment or capital utilization, uncorrelated with \(\theta_t\), they will be reflected through the production function relation, and the SR will be uncorrelated with them, provided we have CRS and perfect competition.
Market Power, Cost based shares

\[ Q = F(K, L) \]

\[ \text{Max } L, K \quad p(Q)Q - wL - rK \]

\[ w = [p + Qp'(Q)] \frac{dQ}{dL}; \quad r = [p + Qp'(Q)] \frac{dQ}{dK} \]

\[ \frac{dQ}{dL} = \frac{w}{p \left[ 1 + \frac{p'}{p} Q \right]} = \frac{w}{p(1 - \varepsilon^{-1})} \]

\[ \frac{dQ}{dK} = \frac{r}{p \left[ 1 + \frac{p'}{p} Q \right]} = \frac{r}{p(1 - \varepsilon^{-1})} \]

Markup and Elasticity:

Markup is \( \mu \) and elasticity is \( \varepsilon = \left( \frac{dP(Q)}{dQ} \frac{Q}{P} \right)^{-1} \). Then

Marginal cost: \( \frac{w}{Q_L} = p(1 - \varepsilon^{-1}) = \frac{r}{Q_K} \)

\[ \mu = \frac{p}{MC} = \frac{p}{r} = \frac{p}{w} = (1 - \varepsilon^{-1})^{-1} \]

\( \varepsilon = \infty \) is perfect competition: the lower the demand elasticity, the higher the markup
COST SHARES: From CRS

\[ Q = \frac{dQ}{dL} L + \frac{dQ}{dK} K = \frac{wL + rK}{p(1 - \varepsilon^{-1})} \]

\[ pQ = \frac{wL + rK}{(1 - \varepsilon^{-1})} > wL + rK \]

\[ \frac{Q}{wL + rK} = \frac{1}{p(1 - \varepsilon^{-1})}, \]

Cost Share :

\[ \frac{wQ}{wL + rK} = \frac{w}{p(1 - \varepsilon^{-1})} \]

\[ \frac{Q}{wL + rK} = \frac{1}{p(1 - \varepsilon^{-1})}, \]

Cost Share :

\[ \frac{rQ}{wL + rK} = \frac{r}{p(1 - \varepsilon^{-1})} \]

So it follows that (Cost Shares)

\[ \frac{wQ}{wL + rK} = \frac{w}{p(1 - \varepsilon^{-1})} = \frac{dQ}{dL} \geq \frac{w}{p} \]

\[ \frac{rQ}{wL + rK} = \frac{r}{p(1 - \varepsilon^{-1})} = \frac{dQ}{dK} \geq \frac{r}{p} \]
Correction for Market Power:
\[
\frac{dQ}{Q} = \left( \frac{K}{Q} \frac{\partial F}{\partial K} \right) \frac{dK}{K} + \left( \frac{L}{Q} \frac{\partial F}{\partial L} \right) \frac{dL}{L} + \theta
\]

Now define
\[
\frac{\partial F}{\partial L} = \alpha \frac{Q}{L} = \frac{wQ}{wL + rK}
\]

1 = \frac{\partial F}{\partial L} \frac{L}{Q} + \frac{\partial F}{\partial K} \frac{K}{Q} \quad \text{from CRS}

1 - \alpha = \frac{\partial F}{\partial K} \frac{K}{Q}

So (COST BASED SHARES):
\[
\alpha = \frac{\partial F}{\partial L} \frac{L}{Q} = \frac{wL}{wL + rK}
\]

1 - \alpha = \frac{\partial F}{\partial K} \frac{K}{Q} = \frac{rK}{wL + rK}

\[
\frac{dQ}{Q} = \alpha \frac{dL}{L} + (1 - \alpha) \frac{dK}{K} + \theta
\]
These $\alpha'$s are different from Solow’s shares $\alpha_s = \frac{wL}{pQ}$: Since

$$pQ = \frac{wL + rK}{(1 - \epsilon^{-1})}$$

$$\alpha = \frac{wL}{rK + wL} = \frac{wL}{pQ(1 - \epsilon^{-1})} > \frac{wL}{pQ} = \alpha_s$$

$$\alpha_s = \alpha(1 - \epsilon^{-1})$$
Because $\alpha > \alpha_s$, an increase in $L$ does not generate enough of an increase in $Q$ under Solow’s accounting with market power:

$$\frac{dQ}{Q} = \alpha \frac{dL}{L} + (1 - \alpha) \frac{dK}{K} + \theta$$

$$= \frac{\alpha_s}{1 - \varepsilon^{-1}} \frac{dL}{L} + \left(1 - \frac{\alpha_s}{1 - \varepsilon^{-1}}\right) \frac{dK}{K} + \theta$$

Since markup, $\mu = (1 - \varepsilon^{-1})^{-1}$

$$\frac{dQ}{Q} = \alpha_s \mu \frac{dL}{L} + (1 - \alpha_s \mu) \frac{dK}{K} + \theta$$

But

$$SR = \theta = \frac{dQ}{Q} - \alpha_s \frac{dL}{L} - (1 - \alpha_s) \frac{dK}{K}$$

$$= (\mu - 1)\alpha_s \left(\frac{dL}{L} - \frac{dK}{K}\right)$$

Higher $\frac{L}{K}$ increases $SR$ under Solow accounting, but not under cost shares.
Increasing Returns

\[ Q = F(K, L) \]
\[ \delta Q = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L, \quad \delta > 1 \]
\[ \delta Q = rK + wL, \]

Note: under IR There are losses.

\[ Q = \frac{rK + wL}{\delta} : \quad \frac{Q}{rK + wL} = \frac{1}{\delta} \]
\[ w \frac{\delta Q}{rK + wL} = w = \frac{\partial F}{\partial L} ; \quad r \frac{\delta Q}{rK + wL} = r = \frac{\partial F}{\partial K} ; \]
COST BASED SHARES UNDER IR:

Define $\alpha$ :

$$\frac{\partial F}{\partial L} = \frac{\delta wQ}{rK + wL} = \alpha \delta \frac{Q}{L};$$

$$\alpha = \frac{wL}{rK + wL}; \quad (1 - \alpha) = \frac{rK}{rK + wL}$$

So

$$dQ = \frac{\partial F}{\partial K} dK + \frac{\partial F}{\partial L} dL$$

$$\frac{dQ}{Q} = \left(\frac{K}{Q} \frac{\partial F}{\partial K} \frac{dK}{K} + \frac{L}{Q} \frac{\partial F}{\partial L} \frac{dL}{L}\right) + \theta$$

Since

$$r = \frac{\partial F}{\partial K} = \frac{r\delta Q}{rK + wL}; \quad w = \frac{\partial F}{\partial L} = \frac{w\delta Q}{rK + wL}$$

$$\frac{dQ}{Q} = \left(\frac{K}{Q} \frac{\delta rQ}{rK + wL} \frac{dK}{K} + \frac{L}{Q} \frac{\delta wQ}{rK + wL} \frac{dL}{L}\right) + \theta$$

$$\frac{dQ}{Q} = \delta \left[\left(\frac{rK}{rK + wL} \frac{dK}{K} + \frac{wL}{rK + wL} \frac{dL}{L}\right)\right] + \theta$$

$$\frac{dQ}{Q} = \delta \left[(1 - \alpha) \frac{dK}{K} + \alpha \frac{dL}{L}\right] + \theta$$
Then, $SR$ based on cost shares:

$$\frac{dQ}{Q} = \delta \left[ (1 - \alpha) \frac{dK}{K} + \alpha \frac{dL}{L} \right] + \theta$$

$$SR = \frac{dQ}{Q} - \alpha \frac{dL}{L} - (1 - \alpha) \frac{dK}{K}$$

$$= \theta + (\delta - 1)\alpha \frac{dL}{L} + (\delta - 1)(1 - \alpha) \frac{dK}{K}$$

So $SR$ on cost based shares now correlated with changes in labor and capital.
Modelling difficulties

Estimating problems:

\[ Q_t = A_t F(K_t, L_t) \]

a) Variations in capacity utilization of capital biases coefficient of \( L \) upwards with short-run, high frequency data.

b) In general business cycles variations can interfere.

c) Suppose \( A_t = A_{t-1} + \varepsilon_t \), so \( A_t \) is serially correlated. High \( A_t \) today implies high \( A_{t+1} \). But high \( A_t \) causes high savings and high investment, and therefore high \( K_{t+1} \), which implies that \( \text{corr}(K_{t+1}, \varepsilon_{t+1}) > 0 \). This also biases estimates. Differencing does not help unless \( \varepsilon_t \) is a random walk.

(Benhabib-Jovanovic, AER, 1995).
Modelling:

\[ Y_t = v_t F(K_t, A_t L_t) \]
\[ K_{t+1} = (1 - \delta)K_t + Y_t - C_t \]
\[ C_t = (1 - s)Y_t, \quad L_t = 1 \]
\[ A_t = A_0 \gamma^t \]
\[ v_t \sim D[\alpha, \beta], \quad iid, \quad E(v_t) = \bar{v} \]
\[ \frac{Y_t}{A_t L_t} = y_t = v_t f(k_t) \]
\[ K_{t+1} = (1 - \delta)K_t + (1 - s)v_t F(K_t, A_t L_t) \]
\[ \frac{K_{t+1}}{A_t L_t} = (1 - \delta) \frac{K_t}{A_t L_t} + (1 - s)v_t f(k_t) \]
\[ \frac{K_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1} L_{t+1}}{A_t L_t} = \gamma k_{t+1} = (1 - \delta)k_t + (1 - s)v_t f(k_t) \]
Steady State:

\[ \gamma \bar{k} = (1 - \delta)\bar{k} + (1 - s)\bar{v}f(\bar{k}) \]

\[ \bar{y} = \bar{v}f(\bar{k}) \]

\[ \hat{k}_t = \frac{k_t - \bar{k}}{\bar{k}}; \quad \hat{v}_t = \frac{v_t - \bar{v}}{\bar{v}}; \quad \hat{y}_t = \frac{y_t - \bar{y}}{\bar{y}} \]
Linearization

\[ \hat{k}_{t+1} = a_1 \hat{k}_t + c_1 \hat{v}_t \]
\[ \hat{y}_t = b_2 \hat{k}_t + \hat{v}_t; \quad \hat{k}_t = (b_2)^{-1}(\hat{y}_t - \hat{v}_t) \]
\[ (b_2)^{-1}(\hat{y}_{t+1} - \hat{v}_{t+1}) = a_1 (b_2)^{-1}(\hat{y}_t - \hat{v}_t) + c_1 \hat{v}_t \]
\[ \hat{y}_{t+1} = a_1 \hat{y}_t + (b_2 c_1 - a_1) \hat{v}_t + \hat{v}_{t+1} \]
\[ \hat{y}_{t+1} = a_1 \hat{y}_t + \varepsilon_t \]
\[ \varepsilon_t = (b_2 c_1 - a_1) \hat{v}_t + \hat{v}_{t+1} \]

Now, \( ln(\hat{y}_{t+1}) = ln(\frac{y_t}{\bar{y}}) \approx \hat{y}_t \) (expanding around 1)

\[ y_t = \frac{Y_t}{A_0 \gamma^t} \]
\[ \hat{y}_{t+1} = ln(\frac{y_{t+1}}{\bar{y}}) \]
\[ \hat{y}_{t+1} = ln(Y_{t+1}) - (t + 1) ln(\gamma) - ln(A_0 \bar{y}) \]
\[ = a ln(Y_t) - a t ln(\gamma) - a ln(A_0 \bar{y}) + \varepsilon_t \]
\[ \tilde{y}_{t+1} = ln(Y_{t+1}) = a ln(Y_t) + bt + c + \varepsilon_t \]

But \( \varepsilon_t \) is serially correlated.
So, Using Cobb-Douglas with $\alpha$, $1 - \alpha$ shares

\[
\ln(Y_{t+1}) = a_0 + \alpha \ln(K_t) + (1 - \alpha) \ln(L_t) + a_1 t + a_2 v_t
\]

\[
v_t = a_4 v_{t-1} + c_4 t + e_t
\]

\[
\tilde{y}_t = a_1 \tilde{y}_{t-1} + bt + (b_2 c_1 - a_1) \hat{v}_{t-1} + \hat{v}_t
\]

\[
\hat{v}_t = a_4 \hat{v}_{t-1} + c_4 t + \epsilon_{t-1}
\]