0.1 Lucas 1988 JME- *On the Mechanics of Development*

\[
\max_{c(t)} \int_0^\infty \left( \frac{c^{1-\sigma} - 1}{1 - \sigma} \right) N(t) e^{-\rho t} dt
\]

subject to:

\[
Nc + \dot{K} = AK^\beta N^{1-\beta}, \quad \frac{\dot{A}}{A} = \mu
\]

\[
H = N \left( \frac{c^{1-\sigma} - 1}{1 - \sigma} \right) + \theta \left( AK^\beta N^{1-\beta} - Nc \right)
\]

where \(c\) is consumption, \(N\) is labor force that grows at the rate \(\lambda\), \(K\) is physical capital, \(\rho\) is a positive discount factor, \(A\) and \(\delta\) are positive technology parameters, \(\beta\) is the share of capital, and \(\sigma\) is the inverse of the intertemporal elasticity of substitution.

\[
H = N \left( \frac{c^{1-\sigma} - 1}{1 - \sigma} \right) + \theta \left( AK^\beta N^{1-\beta} - Nc \right)
\]

FOC

\[
c^{-\sigma} = \theta \quad \dot{\theta} = \theta \left( \rho - \beta AN^{1-\beta} K^{\beta-1} \right)
\]

\[
\lim_{t \to \infty} e^{-\rho t} \theta K = 0
\]

Now, along a balanced growth path where \(\frac{\dot{c}}{c} \equiv \kappa\):

\[
\dot{\theta} = -\sigma c^{-\sigma-1} c^{-\sigma} \frac{\dot{c}}{c} = -\sigma c^{-\sigma} = -\sigma \kappa
\]

But from FOC,

\[
\dot{\theta} = \left( \rho - \beta AN^{1-\beta} K^{\beta-1} \right)
\]

\[
\dot{\theta} + \rho = \beta AN^{1-\beta} K^{\beta-1}; \quad MPK = \rho + \sigma \kappa
\]

If production is Cobb-Douglas:

\[
MPK = \beta \frac{F(K, N)}{K} = \beta \frac{AN^{1-\beta} K^\beta}{K}
\]

\[
\frac{Nc + \dot{K}}{K} = A \left( \frac{N}{K} \right)^{1-\beta} = \beta^{-1} MPK = \frac{\rho + \sigma \kappa}{\beta}
\]

So, if \(\frac{\dot{K}}{K}\) is constant along a balanced growth path, so is \(\frac{Nc}{K}\). This implies that

\[
\frac{\dot{N}}{N} + \frac{\dot{c}}{c} = \frac{\dot{K}}{K} = \lambda + \kappa
\]
(λ + κ is capital growth, and also output growth defined as the growth of \( Nc \).)

Now some Algebra: On a BGP

\[
\frac{dA}{dN} \left( \frac{N}{K} \right)^{1-\beta} = \dot{A} \left( \frac{N}{K} \right)^{1-\beta} + A (1 - \beta) \left( \frac{N}{K} \right)^{-\beta} \left( \frac{NK - KN}{K^2} \right)
\]

: On a BGP

\[
\frac{dA}{dN} \left( \frac{N}{K} \right)^{1-\beta} = \dot{A} + (1 - \beta) \left( \frac{\dot{N}}{N} - \frac{\dot{K}}{K} \right) = \mu + (1 - \beta) (\lambda - \kappa - \lambda)
\]

\[
\frac{\mu}{1 - \beta} = \kappa
\]

Thus, \( \kappa \) is independent of \( \rho \) or \( \sigma \). Let the savings rate

\[
s = \frac{\dot{K}}{Nc + K}
\]

\[
\frac{Nc + \dot{K}}{K} = \frac{\rho + \sigma \kappa}{\beta}
\]

\[
s^{-1} = \left( \frac{Nc + \dot{K}}{K} \right) \frac{K}{\beta} = \frac{\rho + \sigma \kappa}{\beta (\kappa + \lambda)}
\]

\[
s = \frac{\beta (\kappa + \lambda)}{\rho + \sigma \kappa}
\]

Saving rate does not affect \( \kappa \).

Checking Transversality along a balanced growth path:

\[
\frac{\dot{K}}{K} = \kappa + \lambda; \quad K(t) = K(0) e^{(\kappa + \lambda)t}
\]

\[
\dot{\theta} = -\sigma \kappa \theta(t) = \theta(0) e^{-\sigma \kappa t}
\]

\[
\lim_{t \to \infty} e^{-\rho t} K(0) \theta(0) e^{(\kappa + \lambda - \sigma \kappa)} = 0
\]

\[
i f \sigma \kappa + \rho > \kappa + \lambda
\]

\[
\kappa (\sigma - 1) + \rho - \lambda > 0
\]
CALIBRATION:

\[
\begin{align*}
\lambda &= 0.013, \quad (1 - \beta) = 0.75 \quad s = 0.1 \\
\kappa + \lambda &= (0.024, \ 0.029), \ \text{say} \ 0.027 \quad \kappa = 0.014 \\
\mu &= \kappa (1 - \beta) = 0.0105
\end{align*}
\]

Then, from \( s = \frac{\beta (\kappa + \lambda)}{\rho + \sigma \kappa} \),

\[
\rho + (0.014) \sigma = 0.0675 \quad \text{if} \quad \sigma = 1, \ \rho = 0.0535
\]

Problem that remains: How to account for persistent growth rate differentials between countries? Also, technology, or the production function implies that per capita output and capital growth are related by, from the production function,

\[
g_{yt} - \beta g_{kt} = \mu
\]

which does not seem uniform across countries. Introducing another factor, like knowledge or human capital may help, if combined with the removal of diminishing returns in its production.

Finally, a good share of persistent growth is accounted by \( \mu \). What is it?

0.1.1 Introduce Human Capital

\[
N = \int_0^{\infty} N (h) \, dh
\]

Effective labor force, with fraction \( u(h) \) working in production:

\[
N^c = \int_0^{\infty} N (h) \, u(h) \, dh
\]

Production and wage:

\[
Y = F (N^c, K), \quad Wage = F_N h \\
Earnings = F_N h u (h)
\]

Average skill:

\[
h_a = \frac{\int_0^{\infty} N (h) \, h \, dh}{\int_0^{\infty} N (h) \, dh}
\]

If all workers have same skill, and choose same \( u \), \( N^c = uhN \).

Production, with external effect:

\[
N (t) c (t) + \dot{K} (t) = AK^\beta \left [ uh (t) N (t) \right ]^{1 - \beta} (h_a (t)) ^\gamma
\]

Human capital accumulation:

\[
\dot{h} = \delta h \left ( 1 - u \right )
\]
PLANNER: Directly to Hamiltonian

\[
\begin{align*}
\max_{c,u} & \left( c^{1-\sigma} - 1 \right) \\
+ & \theta_1 \left( AK^{\beta} [uh(t)N(t)]^{1-\beta} (h_a(t))^{\gamma} - Nc \right) \\
+ & \theta_2 (\delta h (1-u))
\end{align*}
\]

FOC

\[
\begin{align*}
\dot{\theta}_1 &= c^{-\sigma} \\
\theta_2 \delta h &= \theta_1 (1-\beta) AK^{\beta} [uh(t)N(t)]^{1-\beta} (h_a(t))^{\gamma} (hN) \\
\dot{\theta}_1 &= \rho \theta_1 - \theta_1 \beta AK^{\beta-1} [uh(t)N(t)]^{1-\beta} (h_a(t))^{\gamma} \\
\dot{\theta}_2 &= \rho \theta_2 - \theta_1 AK^{\beta} [uN(t)]^{1-\beta} h(t)^{-\beta+\gamma} (1-\beta-\gamma) - \theta_2 \delta (1-u)
\end{align*}
\]

For the equilibrium solution however:

\[
\dot{\theta}_2 = \rho \theta_2 - \theta_1 AK^{\beta} [uN(t)]^{1-\beta} h(t)^{-\beta+\gamma} (1-\beta) - \theta_2 \delta (1-u)
\]

BGP:

Using

\[
\frac{\dot{\theta}_1}{\theta_1} = -\kappa \sigma
\]

as before

\[
\frac{Nc + \dot{K}}{K} = AK^{\beta-1} [uh(t)N(t)]^{1-\beta} (h_a(t))^{\gamma} = \frac{MPK}{\beta} = \frac{\rho + \sigma \kappa}{\beta}
\]

Again, if \( \frac{\dot{K}}{K} \) is constant, so is \( \frac{Nc}{K} \), and \( s \) is also constant.

Let

\[
v = \frac{\dot{h}}{h} = \delta (1-u)
\]

Differentiating (*) above with \( A \) fixed,

\[
(\beta - 1) \frac{K}{K} + (1-\beta+\gamma) \frac{\dot{h}}{h} + (1-\beta) \frac{\dot{N}}{N} = 0 \\
(\beta - 1) (\kappa + \lambda) + (1-\beta+\gamma) v - (\beta - 1) \lambda = 0 \\
\left( -\frac{(1-\beta+\gamma)}{(\beta - 1)} \right) v = \kappa
\]
NOTE: If $\gamma = 0$, $\kappa = v$. Note also that $(1 - \beta + \gamma) v$ is like $\mu$, the rate of technical change, in the previous model, where we had $\kappa = \frac{\mu}{1 - \beta}$.

DERIVATION OF $v$ (assuming at first we are working on the efficient path so the planner takes $\gamma$ into account.)

Differentiating FOC wrt $u$

\[
\begin{align*}
\theta_1 (1 - \beta) AK^\beta [uh(t)N(t)]^{-\beta} (h_a(t))^\gamma (hN) &= \theta_2 h \\
\frac{\dot{\theta}_1}{\theta_1} + \beta \frac{\dot{K}}{K} + (1 - \beta) \frac{\dot{N}}{N} + (1 - \beta + \gamma) \frac{\dot{h}}{h} &= \frac{\dot{\theta}_2}{\theta_2} + \frac{\dot{h}}{h} \\
-\sigma \kappa + \beta (\kappa + \lambda) + (1 - \beta) \lambda + (\gamma - \beta) v &= \frac{\dot{\theta}_2}{\theta_2} \\
(\beta - \sigma) \kappa + \lambda + (\gamma - \beta) v &= \frac{\dot{\theta}_2}{\theta_2}
\end{align*}
\]

Now lets get $\left(\frac{\dot{\theta}_2}{\theta_2}\right)$ from FOC.

\[
\begin{align*}
\dot{\theta}_2 &= \rho \theta_2 - \theta_1 AK^\beta [uh(t)N(t)]^{1-\beta} h(t)^{\gamma-\beta} (1 - \beta - \gamma) \\
&- \theta_2 \delta (1 - u)
\end{align*}
\]

To get $\left(\frac{\dot{\theta}_2}{\theta_2}\right)$, we must eliminate middle term with $\theta_1$. But from FOC wrt $u$,

\[
\theta_1 (1 - \beta) AK^\beta [uh(t)N(t)]^{-\beta} (h_a(t))^\gamma (hN) = \theta_2 h
\]

Multiplying both sides by $\left(\frac{(1-\beta+\gamma)}{1-\beta}\right)$

\[
= \frac{\theta_2 \delta (1 - \beta + \gamma) u}{1 - \beta}
\]

Substituting into FOC for the state equation in $\theta_2$

\[
\begin{align*}
\dot{\theta}_2 &= \rho \theta_2 - \frac{\theta_2 \delta (1 - \beta + \gamma) u}{1 - \beta} - \theta_2 \delta (1 - u) \\
\frac{\dot{\theta}_2}{\theta_2} &= \rho - \delta \left(1 + \frac{\gamma}{1-\beta}\right) u - \delta (1 - u) \\
&= \rho - \delta u - \frac{\gamma \delta u}{1 - \beta} - \delta (1 - u) \\
&= \rho - \frac{\gamma \delta u}{1 - \beta} - \delta
\end{align*}
\]

But since

\[
v = \delta (1 - u) = \delta - \delta u
\]

\[
\frac{\dot{\theta}_2}{\theta_2} = \rho + \left(\frac{\gamma(v - \delta)}{1 - \beta}\right) - \delta
\]
Now plugging this into the equation where we had differentiated the FOC wrt \( u \), which gave us the other equation for \( \frac{\dot{\theta}_2}{\theta_2} \):

\[
(\beta - \sigma) \kappa + \lambda + (\gamma - \beta) v = \frac{\dot{\theta}_2}{\theta_2}
\]

\[
(\beta - \sigma) \kappa + \lambda + (\gamma - \beta) v = \rho + \left( \frac{\gamma (v - \delta)}{1 - \beta} \right) - \delta
\]

and solving:

\[
v^* = -\sigma^{-1} \left[ \frac{(1 - \beta)(\rho - \delta - \lambda) - \delta \gamma}{(1 - \beta + \gamma)} \right]
\]

\[
= -\sigma^{-1} \left[ \frac{(1 - \beta)(\rho - \lambda)}{(1 - \beta + \gamma)} - \frac{(1 - \beta) \delta + \gamma \delta}{(1 - \beta + \gamma)} \right]
\]

\[
= -\sigma^{-1} \left[ \frac{(1 - \beta)(\rho - \lambda)}{(1 - \beta + \gamma)} - \delta \right]
\]

Note: if \( \gamma = 0 \), \( v^* = \sigma^{-1}(\delta + \lambda - \rho) \).

However, on the competitive equilibrium path, agents take \( (h_a)^\gamma \) as exogenous:

\[
\dot{\theta}_2 = \rho \theta_2 - \frac{\theta_2 \delta (1 - \beta) u}{1 - \beta} - \theta_2 \delta (1 - u)
\]

\[
\frac{\dot{\theta}_2}{\theta_2} = \rho - \delta
\]

so

\[
v = - \frac{[(1 - \beta)(\rho - \delta - \lambda) - \delta \gamma]}{\sigma (1 - \beta + \gamma) - \delta}
\]

NOTE:

If \( \gamma = 0, v^* = \sigma^{-1}(\delta + \lambda - \rho) \).

\[
v^* - v = \frac{\gamma}{(1 - \beta + \gamma)} (\rho - \lambda)
\]

Computing Balanced Growth Path:

\[
\frac{Nc + \dot{K}}{K} = AK^{\beta - 1} \left[ u h(t) N(t) \right]^{1-\beta} (h_a(t))^{\gamma} = \frac{\rho + \sigma \kappa}{\beta}
\]

\[
\rho + \sigma \kappa
\]

\[
= \beta A \left( K(0) e^{(\kappa + \lambda)t} \right)^{\beta - 1} \left[ u h(0) e^{\beta t} N(0) e^{\lambda t} \right]^{1-\beta} (h(0) e^{\beta t})^{\gamma}
\]

\[
= \beta A (K(0) e^{\beta t})^{\beta - 1} [uN(0)]^{1-\beta} (h(0) e^{\beta t})^{(1-\beta + \gamma)}
\]

\[
= \beta A (K(0))^{\beta - 1} [uN(0)]^{1-\beta} (h(0))^{(1-\beta + \gamma)} e^{\kappa(\beta-1)t} e^{(1-\beta + \gamma)vt}
\]
But, on a BGP we have
\[(1 - \beta + \gamma) v = (1 - \beta) \kappa\]
so \(e^{(\kappa(\beta-1))t} e^{(1-\beta+\gamma)vt} = e^0 = 1\) and
\[
\beta A (K(0))^{\beta-1} [uN(0)]^{1-\beta} (h(0))^{(1-\beta+\gamma)} = \rho + \sigma \kappa
\]
which defines the BGP relation. Let initial conditions correspond to BGP levels, 
\[K(0) = z_1, h(0) = z_2,\]
\[
\beta A (z_1)^{\beta-1} [uN(0)]^{1-\beta} (z_2)^{(1-\beta+\gamma)} = \rho + \sigma \kappa
\]
Then
\[
z_1 = K(t) e^{-(\kappa+\lambda)t}
\]
\[
z_2 = h(t) e^{-vt}
\]
along the BGP. Can draw graph of \(z_1\) against \(z_2\), (increasing). Position will depend on \(v, \kappa, \rho\). A higher \(v\), will imply a higher \(\kappa\), and shift the curve to the right, so for given \(z_1\), we would get a higher \(z_2\).

Calibration:
Denison who gives output or capital growth at roughly 0.027 = \(\kappa + \lambda\)
\[
\lambda = 0.013, \kappa = 0.014,
\]
\[
\beta = 0.25, \delta = 0.05,
\]
and \(v = 0.009\) (from Denison)
\[
v = \delta(1 - u) \rightarrow u = 0.82
\]
From \(\frac{(1-\beta+\gamma)}{(1-\beta)}v = \kappa\), we get
\[
\gamma = 0.417
\]
which is enormous\(^1\). As before we get \(\rho + \sigma \kappa = .0675\). Also Lucas computes, from \(v^* = -\sigma^{-1} \left[ \frac{(1-\beta)(\rho-\lambda)}{(1-\beta+\gamma)} - \delta \right] \)
\[
\begin{array}{ccc}
\sigma & v^* & u^* & \kappa^* \\
1 & 0.24 & 0.52 & 0.037 \\
2 & 0.016 & 0.68 & 0.025 \\
3 & 0.014 & 0.72 & 0.022 \\
\end{array}
\]
Note that as a result, \(u\) is too high. Optimally, we need more time in education. This model explains the US data not much better than Solow’s model: by choosing exogenous technical change \(\mu\), Solow’s model does well, but the Lucas model can be consistent with permanent differentials in income and growth rates (note that in the model with human capital \(v\) depends on preference parameters \(\rho, \sigma\) across countries.

\(^1\)If \(\gamma = 0, \kappa = v = v^* = \sigma^{-1} |\delta - (\rho - \lambda)| = .0014, u = .72\)

But if Denison is right that human capital grows at 0.009, then things like on the job training would be 0.005, to add to a total 0.014.

Also we get \(\delta\) from \(v = \frac{[(1-\beta)(\rho-\delta-\lambda)-\delta\kappa]}{\sigma(1-\beta+\gamma)}\).