Basic Case (no \( \tau \) for simplicity, or \( r \) net of \( \tau \))

Value:

\[
V(w(s, t))
= \max_{c, \omega} \int_t^\infty e^{(\theta + p)(t - v)} \left( \begin{array}{c}
\ln c(s, v) \\
+p\phi((1 - b)\omega(s, v))
\end{array} \right) dv
\]

subject to

\[
\frac{dw(s, t)}{dt} = (r + p)w(s, t) - p\omega - c(s, t)
\]

\[
V(w(s, t))
= \max_{c, \omega} \int_t^\infty e^{(\theta + p)(t - v)} \left( \begin{array}{c}
\ln \left( (r + p)w(s, t) - p\omega \right) \\
-\dot{w}(s, t) \\
+p\chi \ln((1 - b)\omega(s, v))
\end{array} \right) dv
\]
FOC:

Euler wrt $w$:

$$e^{(\theta+p)(t-v)}c^{-1}((r+p)) = \frac{d(-e^{(\theta+p)(t-v)}c^{-1})}{dv}$$

$$= (\theta + p)e^{(\theta+p)(t-v)}c^{-1} + e^{(\theta+p)(t-v)}c^{-2}\dot{c}$$

$$\frac{\dot{c}}{c} = (r+p) - (\theta + p) = (r - \theta)$$

control wrt $\omega$

$$pe^{(\theta+p)(t-v)}c^{-1} = \chi pe^{(\theta+p)(t-v)}\omega^{-1}$$

$$\omega = \chi c$$

$$u' = c^{-1} = \phi'$$

If $\phi((1-b)\omega(s,v))$ is not convex, FOC won’t work: do proper maximum wrt control $\omega$. 
Alternative Hamiltonian format:

\[ H = \ln(c) + p\phi(\omega(s,v)) + \lambda((r + p)w - p\omega - c) \]

Maximizing wrt \( c \):

\[ c^{-1} = \lambda \]

maximizing wrt \( \omega \), where

\[ \phi(\omega(s,v)) = \chi \ln((1 - b)\omega(s,v)) \]

\[ \phi' = \chi \omega^{-1} = \lambda = c^{-1} = u'(c) \]

\[ \chi c = \omega \]

and

\[ \dot{\lambda} = (\theta + p)\lambda - (r + p)\lambda = (\theta - r)\lambda \]

\[ \dot{c} = (r - \theta)c \]

Solving for the consumption function

First order conditions include

\[ \omega(s,t) = \chi c(s,t) \]

\[ \dot{c}(s,t) = (r - \theta)c(s,t) \]
The aggregate dynamics for the agent can then be written as:

\[ \dot{w}(t,s) = (r + p)w(t,s) - (p\chi + 1)c(s,v) \]

Postulating \( c = \eta w \), after some algebra,

\[ \frac{dw(s,t)}{dt} = ((r + p) - \eta(p\chi + 1))w(s,v) \]

So that, equating the growth rate equations for \( \frac{\dot{c}}{c} \) and \( \frac{\dot{w}}{w} \) we verify that in fact

\[ c = \eta w, \quad \eta = \frac{p + \theta}{p\chi + 1} \]

and

\[ \omega = \chi c = \chi\eta w \]
Case with non convex bequest function

Consider

\[ \chi \ln w(v) \]

for \( 0 \leq (1 - b)\omega(s, v) \leq w(v) \)

\[ \phi(\omega(s, v)) = \chi \ln(1 - b)\omega(s, v) \]

for \( w(v) \leq (1 - b)\omega(s, v) \)

where \( \omega(s, v) \leq w(s, v) \). Note therefore that if \( (1 - b)w(s, v) < w(v) \), then \( (1 - b)\omega(s, v) < w(v) \).

Note: We will use Theorem 5 and note 11 in Seierstadt & Sydsaeter, *Optimal Control Theory with Economic Applications* (Arrow’s sufficiency Thm. p. 107), so that the Lagrange multipliers will be continuous functions of time.
Consider maximizing the Hamiltonian

\[
H = \max_{\omega} \ln(c) + p\phi(\omega(s, v)) + \lambda((r + p)w - p\omega - c)
\]

1. If \((1 - b)w(s, v) < \underline{w}(v)\), \((1 - b)\omega(s, v) < \underline{w}(v)\), then \(H\) is declining in \(\omega\) over \(0 \leq (1 - b)\omega(s, v) \leq (1 - b)w(s, v)\) because \(\phi(\omega(s, v)) = \chi \ln \underline{w}(v)\). So for small \(w(s, v)\) the optimal \(\omega\) is zero.

2. If \((1 - b)w(s, v) > \underline{w}(v)\) then there may be, as shown above, an interior local max at

\[
p\phi' \frac{d\omega}{dw} = p\chi \omega = p\lambda = pc^{-1} = pu'(c)
\]

\[
u'(c(s, v)) = \phi'(\omega(s, v))
\]

\[
c^{-1} = \chi \omega
\]

\[
\omega = \chi c
\]

Now show that for \(w(s, v)\) large, there is an \(0 < \hat{\omega} < w(s, v)\) which is a global max.
FIRST SOLVE FOR $\hat{w}$, the switch point:

From smooth pasting at $\hat{w}$ marginal utilities, that is shadow prices $\lambda = c^{-1}$ which are equal to the partial derivative of the value function $V$ with respect to the state $w$, are continuous. Another way to say this is that $\lambda = c^{-1}$ will be continuous on the optimal path and therefore consumption is continuous across $\hat{w}$. Note that on the right $c = \eta \hat{w}$ and $\omega = \chi \eta \hat{w}$. Since $\lambda$ is continuous across $\hat{w}$, and the Hamiltonian is equalized across $\hat{w}$ at the optimal choice of $\omega$ at the point of switching from $\omega = 0$ to $\omega = \eta \chi \hat{w}$:

$$
\ln(\lambda^{-1}) + \chi \ln \hat{w}(t) + \lambda \ln((r + p)\hat{w} - \lambda^{-1})
$$

$$
= \ln(\lambda^{-1}) + \chi \ln((1 - b)\omega) + \lambda((r + p)\hat{w} - \lambda^{-1} - p\omega)
$$
SO

\[ \chi \ln \underline{w}(\hat{t}) = \chi \ln((1 - b)\omega(s, \nu)) - \lambda p \omega \]
\[ = \chi \ln((1 - b)\omega) - c^{-1} p \chi \omega \]

\[ \chi \ln \underline{w}(\hat{t}) = \chi \ln((1 - b)\omega) - (\eta \hat{w})^{-1} p \chi \eta \hat{w} \]

\[ \chi \ln \underline{w}(\hat{t}) = \chi \ln((1 - b)\omega(s, \nu)) - p \chi \]

\[ \ln \underline{w}(\hat{t}) = \ln((1 - b)\chi \eta \hat{w}) - p \]

\[ \ln \underline{w}(\hat{t}) + p = \ln((1 - b)\chi \eta \hat{w}) \]

\[ e^p = \frac{((1 - b)\chi \eta \hat{w})}{\underline{w}(\hat{t})} \]

\[ \hat{w} = \frac{e^p}{(1 - b)\chi \eta \underline{w}(\hat{t})} > \underline{w}(\hat{t}) \]
So for $w > \hat{w}$, we switch to interior $\omega > 0$. To show Hamiltonian is concave for $w < \hat{w}$:

$$H(w, \lambda) = \ln(c) + p\phi(\omega(s, v))$$

$$+ \lambda((r + p)w - p\omega - c)$$

$$c^{-1} = \lambda$$

and at the left of $\hat{w}$, $\omega = 0$, so

$$H(w, \lambda) = \ln(\eta w) + p\chi \ln w(v)$$

$$+ \lambda((r + p)w - \lambda^{-1})$$

which is concave in $w$. 
Now we show it is smooth across $\hat{w}$. From the left at $\hat{w}$:

$$H(w, \lambda) = \ln(\eta w) + p\chi \ln w(v) + \lambda((r + p)w - \lambda^{-1})$$

$$\frac{dH(w, \lambda)}{dw} = w^{-1} + \lambda(r + p)$$

From the right at $\hat{w}$:

$$H(w, \lambda) = \ln(\eta w) + p\chi \ln((1 - b)\eta \chi w) + \lambda((r + p)w - \lambda^{-1} - p\omega)$$

$$H(w, \lambda) = \ln(\eta w) + p\chi \ln((1 - b)\eta \chi w) + \lambda((r + p)w - \lambda^{-1} - p\eta \chi w)$$

$$\frac{dH(w, \lambda)}{dw} = (1 + p\chi)w^{-1} + \lambda(r + p) - p\lambda \chi \eta$$

$$= (1 + p\chi)w^{-1} + \lambda(r + p) - p(\eta w)^{-1} \chi \eta$$

$$= (1 + p\chi)w^{-1} + \lambda(r + p) - pw^{-1} \chi$$

$$= w^{-1} + \lambda(r + p)$$

So the Hamiltonian smooth across $\hat{w}$, since it is also concave for $w > \hat{w}$, the Hamiltonian is globally concave in $w$, so we have the sufficient condition for an
optimum.
Complete solution

Now we turn to compute $\hat{t}$, the switch time for an agent born at 0 with $w(0,0) = w(0)$. We have

$$\dot{c} = c(r - \theta)$$

$$c(t) = c(0)e^{(r-\theta)t}$$

$$\dot{w} = (r + p)w - c(0)e^{(r-\theta)t}$$

$$w(t) = e^{(r+p)t}\left(w(0) + \frac{c(0)}{p + \theta}(e^{-(p+\theta)t} - 1)\right)$$

At switch point $\hat{w}$ at $\hat{t}$

$$\hat{w} = \left(\frac{e^p}{(1-b)\chi\eta} w(\hat{t})\right)$$

$$\hat{w} = e^{(r+p)\hat{t}}\left(w(0) + \frac{c(0)}{p + \theta} (e^{-(p+\theta)\hat{t}} - 1)\right)$$

Store $\hat{w}$. 
\[ \frac{c(t)}{\hat{w}} = \frac{c(0)e^{(r-\theta)\hat{t}}}{e^{(r+p)\hat{t}} \left( w(0) + \frac{c(0)}{p+\theta} (e^{-(p+\theta)\hat{t}} - 1) \right)} = \eta \]

\[ c(0)e^{(r-\theta)\hat{t}} = \eta e^{(r+p)\hat{t}} \left( w(0) + \frac{c(0)}{p+\theta} (e^{-(p+\theta)\hat{t}} - 1) \right) \]

\[ c(0) = \eta e^{(p+\theta)\hat{t}} \left( w(0) + \frac{c(0)}{p+\theta} (e^{-(p+\theta)\hat{t}} - 1) \right) \]

\[ c(0) = \eta \left( e^{(p+\theta)\hat{t}} w(0) - \frac{c(0)}{p+\theta} (e^{(p+\theta)\hat{t}} - 1) \right) \]

\[ c(0) = \frac{\eta e^{(p+\theta)\hat{t}} w(0)}{\left( 1 + \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t}} - 1) \right)} \]

\[ \frac{c(0)}{w(0)} = \eta \frac{e^{(p+\theta)\hat{t}}}{\left( 1 + \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t}} - 1) \right)} \]

\[ = \eta \left( \frac{1}{e^{-(p+\theta)\hat{t}} + \frac{\eta}{p+\theta} (1 - e^{-(p+\theta)\hat{t}})} \right) \]

or more generally for \( t < \hat{t} \)
\[
\frac{c(t)}{w(t)} = \eta \left( \frac{1}{e^{-(p+\theta)(\hat{t}-t)} + \frac{\eta}{p+\theta} \left( 1 - e^{-(p+\theta)(\hat{t}-t)} \right)} \right)
\]
\[ \eta = \frac{p + \theta}{1 + p\chi} \]

Also, the share of annuities
\[ \mu = 1 - \eta \chi = 1 - \frac{(p + \theta)\chi}{1 + p\chi} = \frac{1 - \theta \chi}{1 + p\chi} < 1 \]

If \( \mu \geq 0 \), that is if agents hold annuities rather than life insurance, \( \eta \chi \leq 1 \). So note that
\[
\frac{c(0)}{w(0)} = \eta \left( \frac{e^{(p+\theta)\hat{t}}}{1 + \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t}} - 1)} \right)
\]
\[
= \eta \left( \frac{1}{e^{-(p+\theta)\hat{t}} + \frac{\eta}{p+\theta} (1 - e^{-(p+\theta)\hat{t}})} \right)
\]
\[
= \eta \left( \frac{1}{e^{-(p+\theta)\hat{t}} + \frac{1}{1+p\chi} (1 - e^{-(p+\theta)\hat{t}})} \right) > \eta = \frac{c(\hat{t})}{\hat{w}}
\]

So \( \frac{c(0)}{w(0)} \) declines to \( \frac{c(\hat{t})}{\hat{w}} = \eta \) at \( \hat{t} \)

So \( \frac{c(0)}{w(0)} \) declines to \( \eta \) at \( \hat{t} \). NB: \( \frac{c(0)}{w(0)} \)
depends on $\hat{t}$. 
We now have

\[ \hat{w} = \left( \frac{e^p}{(1-b)\chi \eta} \right) \hat{w}(t) > w(t) \]

\[ = e^{(r+p)t} \left( w(0) + \frac{c(0)}{p + \theta} (e^{-(p+\theta)t} - 1) \right) > w(t) \]

We have \( \eta = \frac{p+\theta}{p\chi + 1} \) and \( c = \eta w \) for \( t \geq \hat{t} \).

Note that if agents hold annuities rather than life insurance, as shown above, \( \chi \eta \leq 1 \), so we have by construction \( \hat{w}(\hat{t}) \leq w(0) \leq \hat{w} \). As \( \hat{w}(\hat{t}) \to 0 \), so does \( \hat{w} \), and therefore so does \( w(0) \). Then if \( \hat{w}(\hat{t}) = w(0) = \hat{w} \), we must have \( \hat{t} = 0 \). If agents hold life insurance and \( \chi \eta \geq 1 \), then we may still have \( \left( \frac{e^p}{(1-b)\chi \eta} \right) > 1 \). If \( \left( \frac{e^p}{(1-b)\chi \eta} \right) < 1 \), agents must hold life insurance for sure and \( \mu < 0 \) because \( 1 - b \chi \eta > 1 \), and bequests exceed the wealth of the agent at death: there is no one that falls below the threshold.
Substituting for \( \frac{c(0)}{w(0)} \) or \( c(0) \)

\[
\hat{w} = \left( \frac{e^p}{(1 - b)\chi\eta} w(\hat{t}) \right)
\]

\[
= e^{(r+p)\hat{t}} \left( \frac{w(0)}{1 + \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t} - 1})} \right) \frac{1}{p+\theta} (e^{-(p+\theta)\hat{t}} - 1)
\]

\[
\hat{w} = \left( \frac{e^p}{(1 - b)\chi\eta} w(\hat{t}) \right)
\]

\[
= e^{(r+p)\hat{t}} w(0) \left( 1 - \frac{\eta}{p+\theta} \frac{(e^{(p+\theta)\hat{t}} - 1)}{1 + \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t} - 1})} \right)
\]

which should solve for \( \hat{t} \) given \( w(\hat{t}) \). But do we know \( w(\hat{t}) \)? We can pick it instead of picking \( w(0) \). This determines \( \frac{c(0)}{w(0)} \) because we now have \( \hat{t} \). So those with wealth in \( (w(\hat{t}), \hat{w}) \) set \( \omega = 0 \).
Therefore the complete solution for \( t < \hat{t} \) is

\[
w(t) = e^{(r+p)t}w(0) \left( 1 + \frac{c(0)}{w(0)} (p + \theta)^{-1} (e^{-(p+\theta)t} - 1) \right)
\]

\[
\frac{c(0)}{w(0)} = \eta(1 + p\chi) \left( \frac{1}{(1 + p\chi)e^{-(p+\theta)\hat{t}} + (1 - e^{-(p+\theta)\hat{t}})} \right)
\]

\[
\frac{c(t)}{w(t)} = \eta \left( \frac{1}{e^{-(p+\theta)(\hat{t}-t)} + \frac{\eta}{p+\theta} (1 - e^{-(p+\theta)(\hat{t}-t)})} \right)
\]

and \( \hat{t} \) is solved by

\[
\hat{w} = \left( \frac{e^p}{(1 - b)\chi\eta} w(\hat{t}) \right)
\]

\[
= e^{(r+p)\hat{t}}w(0) \left( 1 - \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t}} - 1) \right)
\]

\[
= e^{(r+p)\hat{t}}w(0) \left( 1 + \frac{\eta}{p+\theta} (e^{(p+\theta)\hat{t}} - 1) \right)
\]

where we pick \( w(\hat{t}) \) instead of picking \( w(0) \).
Comparing growth rates:

At $t > \hat{t}$:  \[ \hat{g} = \frac{\dot{w}}{w} = r + p - \eta(1 + p\chi) = r - \theta \]

At $t = \hat{t}$ from the left:  \[ \tilde{g} = \frac{\dot{w}}{w} = r + p - \eta \]

At $t = 0$:  \[ g^0 = \frac{\dot{w}}{w} = r + p - \frac{c(0)}{w(0)} \]

\[ = r + p - \eta \left( \frac{(1 + p\chi)}{(p\chi)e^{(p+\theta)\hat{t}} + 1} \right) \]

Thus $g$ declines

\[ g^0 > \tilde{g} > \hat{g} \]

Also

\[ g^0 - \hat{g} = \eta(p\chi + 1) \left( 1 - \frac{1}{(p\chi)e^{(p+\theta)\hat{t}} + 1} \right) \]

\[ = (p + \theta) \left( \frac{(p\chi)e^{(p+\theta)\hat{t}}}{(p\chi)e^{(p+\theta)\hat{t}} + 1} \right) > 0 \]
However we must now note that the agent and economy growth rates for $t < \hat{t}$ are no longer $g$ and $g'$. First for agents with $w < \hat{w}(t)$, $g$ is non-constant as described above. Second, $g'$ has to be adjusted as well, not only because $g$ is variable for $w < \hat{w}$, but because estate taxes are not collected for $w < \hat{w}$. So we have an approximation which gets better as $\underline{w}(\hat{t}) \to 0$.