0.1 Solow Model

\[ \dot{K} = sF(K, L) - \delta K = sLF(k, 1) - \delta K \]

\[ \frac{\dot{K}}{L} = sf(k) - \delta \frac{\dot{K}}{L} = \frac{\dot{K}}{L} - nk \]

where \[ \frac{\dot{L}}{L} = n \]

\[ \dot{k} = sf(k) - (\delta + n)k \]

Golden Rule

\[ \beta - \text{Convergence (not conditional)}: \]

\[ \gamma = \frac{\dot{k}}{k} = sf(k) - (\delta + n) \]

\[ \frac{d\gamma}{dk} = s \left( \frac{k f'(k) - f(k)}{k^2} \right) = \frac{s}{k} \left( f'(k) - \frac{f(k)}{k} \right) = \frac{s}{k} (MPK - APK) < 0 \]

0.1.1 Technical Progress:

\[ \dot{K} = sF(K, A(t) L) - \delta K \]

\[ \dot{L} = n; \dot{A} = \theta; x(t) = \frac{K(t)}{A(t) L(t)} \]

\[ \dot{x} = sf(x) - (\delta + n + \theta) x \]

\[ \gamma = \frac{\dot{x}}{x} = s \frac{f(x)}{x} - (\delta + n + \theta) \]

0.1.2 The Solow Residual

Production Function (Constant Returns to scale):

\[ Q = AF(K, L) \]
\[
\dot{Q} = F_k \dot{K} + F_L \dot{L} + \dot{A}F \\
\frac{\dot{Q}}{Q} = \left( \frac{KF_k}{Q} \right) \frac{\dot{K}}{K} + \left( \frac{LF_L}{Q} \right) \frac{\dot{L}}{L} + \frac{\dot{A}}{A} \\
\frac{\dot{A}}{A} = \frac{\dot{Q}}{Q} - \left( \frac{KF_k}{Q} \right) \frac{\dot{K}}{K} - \left( \frac{LF_L}{Q} \right) \frac{\dot{L}}{L}
\]

Let

\[
k = \frac{K}{L}, \quad q = \frac{Q}{L} = Af(k) \\
\dot{q} = \dot{A}f + Af'k \\
\frac{\dot{q}}{q} = \frac{\dot{A}}{A} + \left( \frac{Af'(k)}{K} \right) \frac{k}{k} = \frac{\dot{A}}{A} + w_k \frac{\Delta k}{k}
\]

Solow adjusted capital for utilization (multiplied by \(employment \ \frac{Labor \ force}{employment} \)), used non-farm output for \(q\), used capital share of 33% from the data, and set \(A_{1909} = 1\). Then

\[
\frac{\Delta q}{q} = \frac{\Delta A}{A} + w_k \frac{\Delta k}{k}
\]

88% of growth accounted by \(\frac{\Delta A}{A}\), 12% by \(\frac{\Delta k}{k}\).

If there is a random element to productivity, \(\frac{\Delta A}{A} = \mu + \theta_t\)

\[
SR_t = \frac{\dot{A}}{A} = \mu + \theta_t = \frac{\dot{Q}}{Q} - \left( \frac{KF_k}{Q} \right) \frac{\dot{K}}{K} - \left( \frac{LF_L}{Q} \right) \frac{\dot{L}}{L}
\]

Issues: (Denison).

Now, if there are variables in the economy that affect output or employment or capital utilization, uncorrelated with \(\theta_t\), they will be reflected through the production function relation, and the SR will be uncorrelated with them, provided we have CRS and perfect competition.

0.1.3 Market Power

I.

\[
Q = F(K, L)
\]

\[
Max_{L,K} \quad p(Q)Q - wL - rK
\]

\[
w = \left[ p + Qp'(Q) \right] \frac{dQ}{dL}; \quad r = \left[ p + Qp'(Q) \right] \frac{dQ}{dK}
\]

\[
\frac{dQ}{dL} = \frac{w}{p \left[ 1 + \frac{p'}{p}Q \right]} = \frac{w}{p(1 - \varepsilon)}; \quad \frac{dQ}{dK} = \frac{r}{p \left[ 1 + \frac{p'}{p}Q \right]} = \frac{r}{p(1 - \varepsilon)}
\]
But from CRS

\[
Q = \frac{dQ}{dL} L + \frac{dQ}{dK} K
\]

\[
= \frac{wL + rK}{p(1 - \varepsilon)} \cdot \frac{Q}{wL + rK} = \frac{1}{p(1 - \varepsilon)}
\]

\[
pQ = \frac{wL + rK}{(1 - \varepsilon)} > wL + rK
\]

Furthermore

\[
\frac{wQ}{wL + rK} = \frac{w}{p(1 - \varepsilon)} = \frac{dQ}{dL}; \quad \frac{rQ}{wL + rK} = \frac{r}{p(1 - \varepsilon)} = \frac{dQ}{dK}
\]

II.

\[
\frac{dQ}{Q} = \left( \frac{K}{Q} \frac{\partial F}{\partial K} \right) \frac{dK}{K} + \left( \frac{L}{Q} \frac{\partial F}{\partial L} \right) \frac{dL}{L} + \theta
\]

Now define \( \alpha \) :

\[
\frac{\partial F}{\partial L} = \frac{\alpha Q}{L} = \frac{wQ}{wL + rK}; \quad \alpha = \frac{wL}{wL + rK}
\]

From CRS

\[
1 = \frac{\partial F}{\partial L} Q + \frac{\partial F}{\partial K} Q
\]

\[
1 - \alpha = \frac{\partial F}{\partial K} Q
\]

Note: \( \frac{\partial F}{\partial L} > \frac{w}{p} \) because \( \frac{Q}{wL + rK} > \frac{1}{p} \) due to market power.

So (COST BASED SHARES)

\[
\alpha = \frac{\partial F}{\partial L} Q = \frac{wL}{wL + rK}
\]

\[
1 - \alpha = \frac{\partial F}{\partial K} Q = \frac{rK}{wL + rK}
\]

Since

\[
\frac{dQ}{Q} = \left( \frac{K}{Q} \frac{\partial F}{\partial K} \right) \frac{dK}{K} + \left( \frac{L}{Q} \frac{\partial F}{\partial L} \right) \frac{dL}{L} + \theta
\]


\[\text{Markup is } \mu \text{ and elasticity is } \varepsilon = \left( \frac{dP(Q)}{dQ} p \right)^{-1}. \text{ Then}
\]

\[
MC = \text{Marginal cost: } \frac{w}{QL} = p(1 - \varepsilon) = \frac{r}{QK}
\]

\[
\frac{p}{MC} = \frac{p}{QL} = \frac{p}{QL} = (1 - \varepsilon)^{-1}
\]

\( \varepsilon = 1 \) is perfect competition, so the lower the demand elasticity, the lower the markup.
then
\[
\frac{dQ}{Q} = \alpha \frac{dL}{L} + (1 - \alpha) \frac{dK}{K} + \theta
\]

These \( \alpha 's \) are different from Solow’s shares \( \alpha_s = \frac{wL}{rK + wL} \):

\[
\alpha = \frac{wL}{rK + wL} > \frac{wL}{pQ} = \alpha_s
\]

Because \( \alpha > \alpha_s \), an increase in \( L \) does not generate enough of an increase in \( Q \) under Solow’s accounting when there is market power:

\[
\frac{dQ}{Q} = \alpha \frac{dL}{L} + (1 - \alpha) \frac{dK}{K} + \theta
\]

Let \((1 - \varepsilon) = \mu^{-1} \) (\( \mu \) is markup)

\[
\frac{dQ}{Q} = \alpha_s \mu \frac{dL}{L} + (1 - \alpha_s \mu) \frac{dK}{K} + \theta
\]

\[
SR = \theta = \frac{dQ}{Q} - \alpha_s \frac{dL}{L} - (1 - \alpha_s) \frac{dK}{K} = (\mu - 1) \alpha_s \left( \frac{dL}{L} - \frac{dK}{K} \right)
\]

So an increase in \( \frac{L}{K} \) causes an increase in \( SR \) under Solow accounting, but not under cost based shares.

**Increasing Returns**

\[
Q = F(K, L)
\]

\[
\delta Q = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L, \quad \delta > 1
\]

\[
\delta Q = rK + wL,
\]

\[
Q = \frac{rK + wL}{\delta}, \quad \frac{Q}{rK + wL} = \frac{1}{\delta}
\]

\[
\frac{\delta wQ}{rK + wL} = w = \frac{\partial F}{\partial L}; \quad \frac{\delta rQ}{rK + wL} = r = \frac{\partial F}{\partial r};
\]

**COST BASED SHARES:**

Define \( \alpha \):

\[
\frac{\delta wQ}{rK + wL} = \alpha \delta \frac{Q}{L}; \quad \frac{\delta rQ}{rK + wL} = \alpha \delta \frac{Q}{K}
\]

\[
\alpha = \frac{wL}{rK + wL}, \quad (1 - \alpha) = \frac{rK}{rK + wL}
\]
So

\[
\frac{dQ}{Q} = K \frac{\partial F}{\partial K} \frac{dK}{K} + L \frac{\partial F}{\partial L} \frac{dL}{L} + \theta
\]

\[
= K \frac{\delta rQ}{Q} \frac{dK}{K} + L \frac{\delta wQ}{Q} \frac{dL}{L} + \theta
\]

\[
= \delta \left[ \frac{rK}{rK + wL} \frac{dK}{K} + \frac{wL}{rK + wL} \frac{dL}{L} \right] + \theta
\]

\[
SR = \frac{dQ}{Q} - \alpha \frac{dL}{L} - (1 - \alpha) \frac{dK}{K} = (\delta - 1) \alpha \frac{dL}{L} + (\delta - 1) (1 - \alpha) \frac{dK}{K}
\]

So SR now correlated with changes in labor and capital.

**Modelling difficulties:**

**Estimation problems:**

\[Q_t = A_t F(K_t, L_t)\]

a) Variations in capacity utilization of capital biases coefficient of \(L\) upwards with short-run, high frequency data.

b) In general business cycles variations can interfere.

c) Suppose \(A_t = A_{t-1} + \varepsilon_t\), so \(A_t\) is serially correlated. High \(A_t\) today implies high \(A_{t+1}\). But high \(A_t\) causes high savings and high investment, and therefore high \(K_{t+1}\), which implies that \(\text{corr}(K_{t+1}, \varepsilon_{t+1}) > 0\). This also biases estimates. Differencing does not help unless \(\varepsilon_t\) is a random walk. (Benhabib-Jovanovic, AER, 1995).