1. (36 points) The random variable $X$ has a uniform distribution on the interval $[0, \theta]$, where $0 < \theta < \infty$, so that the cumulative distribution function (c.d.f.) of $X$ is given by:

$$F_X(x) = \begin{cases} 
0 & x < 0 \\
 x/\theta & 0 \leq x < \theta \\
1 & \theta \leq x 
\end{cases}.$$ 

A random sample of size $n$ is drawn from $F_X$, $\{X_i\}_{i=1}^n \equiv X_1, \ldots, X_n$.

1. Let $Y_n = \max(\{X_i\}_{i=1}^n)$. Find the mean and variance of $Y_n$.

2. Show that the probability limit of $Y_n$, denoted $\text{plim}(Y_n)$, is equal to $\theta$. Do this by showing that $\lim_{n \to \infty} EY_n = \theta$ and $\lim_{n \to \infty} \text{Var}(Y_n) = 0$.

3. Determine whether or not the random variable $\sqrt{n}(Y_n - \theta)$ has a degenerate asymptotic distribution [recall that a distribution is degenerate if it has zero variance].

4. Let $\overline{X}_n \equiv n^{-1} \sum_{i=1}^n X_i$. Determine the mean and variance of $\overline{X}_n$.

5. Find the asymptotic distribution of $2\overline{X}_n$.

6. Compare the asymptotic variance of $Y_n$ and $2\overline{X}_n$ [that is, the variances of the asymptotic distributions]. If it is possible to rank them, which is smallest?

2. (18 points) Let $X$ and $Y$ be two independent random variables. The random variable $X$ is exponentially distributed with density $f_X(x) = \alpha \exp(-\alpha x)$ for $x > 0$ and $\alpha > 0$. The random variable $Y$ is distributed as a chi-square with $k$ degrees of freedom. The moment generating function of $Y$ is given by $m_Y(t) = (1 - 2t)^{-k/2}$ for $t < .5$.

1. Find the m.g.f. of $X$, $m_X(t)$.

2. Form the random variable $Z = X + Y$. Using $m_X(t)$ and $m_Y(t)$, find the mean and variance of $Z$. 

3. (28 points) In each of two subpopulations the random variable $X$ is distributed according to a power distribution, that is

$$F_{X|\alpha}(x|\alpha) = \begin{cases} 
0 & x < 0 \\
\alpha^x & 0 \leq x < 1, \; \alpha > 0 \\
1 & 1 \leq x
\end{cases}$$

The parameter $\alpha$ takes the value 1 in subpopulation 1 and takes the value 2 in subpopulation 2. The population is evenly divided between the two subpopulations.

1. Find the density of $X$ in the population.

2. A population member can be characterized in terms of their value of $X$ and the subpopulation to which they belong, $d$ [which equals 0 if the individual belongs to subpopulation 1 and equals 1 if the individual belongs to subpopulation 2]. Given knowledge of $d$, find the function $h^*(d)$ that solves:

$$h^*(d) = \underset{h(d)}{\text{arg min}} \, E_{X|d}(X - h(d))^2,$$

where $E_{X|d}(X - h(d))^2$ denotes the expectation of the function $(X - h(d))^2$ taken with respect to the conditional distribution of $X$ given $d$.

3. Reconsider the prediction problem above under the restriction that $h(d)$ be a linear function of $d$. Is the solution the same as above? Why or why not?

4. How much of the total variation in $X$ is “attributable” to variability in the conditional expectations of $X|d$? [Use the Analysis of Variance decomposition.]

4. (18 points) Let $\bar{X}_n$ denote the sample mean of a random sample consisting of $n$ draws from an exponential distribution with parameter $\delta$, i.e., $f_X(x) = \delta \exp(-\delta x)$ for $x > 0$ and $\delta > 0$.

1. Find the asymptotic distribution of $\bar{X}_n$.

2. The maximum likelihood estimate of $\delta$ is given by $\hat{\delta} = \bar{X}_n^{-1}$. Find the asymptotic distribution of $\hat{\delta}$. 

2