Due Date: 4/20/99

Please show all of your work and clearly indicate your final response to each question.

1. A random sample of \( N \) (initially unemployed) labor market entrants are followed from the time they enter the labor market until either (1) they find a job or (2) a 12-month period expires. In the population the length of an unemployment spell (in months) is distributed as a negative exponential, with cumulative distribution function \( F(t) = 1 - \exp(-\alpha t) \) and density \( f(t) = \alpha \exp(-\alpha t), \alpha > 0 \). Denote the observed unemployment durations in the sample by \( (t^*_1, \ldots, t^*_N) \).

1. Let \( p_C \) denote the proportion of the sample for which the observation on the duration of the unemployment spell is right-censored, that is, \( p_C = \frac{\sum_i \chi[t^*_i = 12]}{N} \), where \( \chi[A] \) is the indicator function which takes the value 1 if \( A \) is true and otherwise equals 0. If one exists, define a consistent estimator of \( \alpha \) based only on \( p_C \), and derive its \( \sqrt{N} \) asymptotic variance.

2. Assume that the population median value of unemployment duration is less than 12. Denote the sample median by \( m \). Define a consistent estimator of \( \alpha \) based on \( m \) if one exists (you don’t need to derive the asymptotic variance explicitly). If the population median was greater than 12, would a consistent estimator exist?

3. Define the full-information maximum likelihood estimator of \( \alpha \) using all of the sample observations, and determine the \( \sqrt{N} \) asymptotic variance of your estimator.

4. Imagine that all unemployment spells begin at the same moment in calendar time [normalized to 0], and that you believe that the rate of exit from the unemployment state is a function of demand conditions, which at any instant \( u \) are summarized by the scalar \( s(u) \). Specify the hazard rate for leaving unemployment at duration \( t \) by \( \alpha(t) = \exp(\beta_0 + \beta_1 s(t)) \). Define the maximum likelihood estimator for \( \beta \) using all \( N \) sample observations and the observed function \( s(u), 0 \leq u \leq 12 \).
5. Imagine instead that you believe that the population density of unemployment spells is given by \( f(t|\alpha_j) = \alpha_j \exp(-\alpha_j t) \), where \( j = 1, 2 \), and \( 0 < \alpha_1 < \alpha_2 \). That is, the model specifies the existence of two groups in the population (group membership is not observable by the analyst), each of which has a negative exponential distribution of unemployment spells. Let the population proportion of type one individuals be given by \( \pi_1 \).

1. Write down the log likelihood function for this model using the entire sample. Which parameters are identified?

2. Using the maximum likelihood estimates, describe how you could construct a test of the hypothesis that there only existed one group in the population (as opposed to two).

3. Under the assumption that there exists only one group in the population, an analyst specifies the hazard function as \( h(t) = \exp(\beta_0 + \beta_1 t) \) and obtains maximum likelihood estimates of \( \beta \) using the entire sample described above. Under her specification, she conducts an asymptotically valid test of the null hypothesis that \( \beta_1 = 0 \), and finds that she can not reject it at a reasonable significance level. On the basis of this test result, she claims that there only exists one group in the population and that the population distribution of duration times is negative exponential. Comment on the strengths and weaknesses of her claim.

2. Let \( Y \) be uniform on the interval \([-\theta, \theta]\). You have an i.i.d. sample of \( n \) draws from this distribution, \( \{y_1, ..., y_n\} \).

1. Find the maximum likelihood estimator of \( \theta \), denoted \( \hat{\theta} \). Show that \( \hat{\theta} \) is consistent. Derive its limiting (nondegenerate) distribution (you must determine the correct normalization factor, which is a function of \( n \)).

2. If one exists, define a consistent estimator of \( \theta \) based only on the sample mean, \( \bar{y} \).

3. If one exists, define a consistent estimator of \( \theta \) based only on the sample variance.
3. Agents bid for the right to work for a firm. The bid by agent \( i \) is determined by a draw from a negative exponential distribution with parameter \( \alpha \). Thus the cumulative distribution function of bids is given by

\[
F(b) = 1 - \exp(-\alpha b),
\]

where \( b \) is the bid value and \( \alpha \) is an unknown scalar which is strictly greater than 0.

1. Assume that you have access to a set of bids from one such auction. There were \( N \) bidders for the job, and you have access to each bid made. Write down the maximum likelihood estimator for \( \alpha \) given this sample. Is the m.l.e. an unbiased estimator of \( \alpha \)? Is it consistent? If it is consistent, derive the limiting variance of \( \sqrt{N} (\hat{\alpha} - \alpha) \) when \( N \to \infty \).

2. Assume that you have access only to the lowest bid in the auction - which is the winning bid. Let this value be denoted \( b_{(1)} \), to indicate that it is the first order statistic of the sample. Now however you have access to the winning bid in each of \( A \) independently run auctions, \( b_{(1)}^a, a = 1, \ldots, A \). Each of the \( A \) auctions contained \( N \) bidders. Define the maximum likelihood estimator of \( \alpha \) in this case. Is this estimator consistent for fixed \( N \) as \( A \to \infty \)? Is this estimator consistent for fixed \( A \) as \( N \to \infty \)?