MA2 - Advanced Macroeconomics: Solution to Homework 1

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Exercise n. 1
1) The definition is standard. The competitive equilibrium is not efficient because there is a distortionary tax on capital income, hence the marginal rate of transformation is not equated to the marginal rate of substitution in equilibrium.

2) The government budget constraint is balanced every period, i.e. 
   \[ r(t)\tau(t)a(t) = g(t). \]
Solving the problem of the household through the Hamiltonian, and the static problem of the firm and imposing the equilibrium conditions in the asset market \( a(t) = k(t) \) and in the labor market \( L(t) = 1 \), we arrive at the pair of conditions

\[
\begin{aligned}
\dot{c} &= \frac{1}{\sigma} \left[ (1 - \tau(t)) \left( \alpha k(t)^{\alpha-1} - \delta \right) - \rho \right] \\
\dot{k}(t) &= k(t)^{\alpha} - c(t) - g(t) - \delta k(t).
\end{aligned}
\]

2a) Yes, the \( \dot{c} = 0 \) locus changes, in particular it shifts to the left, as now the steady-state level of capital is given by

\[ k^* = \left[ \frac{\alpha (1 - \tau(t))}{\delta (1 - \tau(t)) + \rho} \right]^{\frac{1}{\alpha - 1}} < \left( \frac{\alpha}{\delta + \rho} \right)^{\frac{1}{\alpha - 1}} \]

which is less than the zero-tax capital stock \( \left( \frac{1}{\delta + \rho} \right)^{\frac{1}{\alpha - 1}} \) as taxes on capital reduces the return and discourage investments. However, also the \( \dot{k} = 0 \) locus shifts down because less resources in the economy are available to private consumption, as some of them are diverted by the government towards public consumption. See Figure 1.

2b) In the long-run, following an increase in \( g \), consumption falls below the original steady-state level, because \( k^* \) is lower and there is less production and income in the economy (distortionary effect of taxes). However, in the short run, the effect could go in the opposite direction if households substitute away from investment into consumption, and consumption might jump up at impact and remain higher for a while. See Figure 2.

3) The question is interesting because now we have distortionary capital taxation, while in class we proved Ricardian equivalence only for lump-sum taxation. The government budget constraint with debt becomes intertemporal

\[ b(t) + r(t)\tau(t)a(t) = g(t) + b(t)\rho(t). \]
Recall that the no-arbitrage condition on the two assets requires \( r(t) [1 - \tau(t)] = r^*(t) \). Call this equilibrium interest rate \( r^*(t) \). This is the rate at which we discount future values of consumption and income. Integrating the intertemporal budget constraint of the government, we obtain

\[ \int_0^\infty e^{-\tau g(t)} dt + b(0) = \int_0^\infty e^{-\tau r(t)k(t)} dt. \]

Let’s move now to the intertemporal budget constraint of the agent

\[ b(t) + \dot{a}(t) + c(t) = r^*(t) [a(t) + b(t)] + w(t) \]

which, in its lifetime counterpart, becomes

\[ \int_0^\infty e^{-\tau r(t)} dt = b(0) + a(0) + \int_0^\infty e^{-\tau r(t)} dt. \]  

Now, observing (1), let us ask: is Ricardian Equivalence violated in this case? Does the timing of taxation affect the total amount of resources available? The answer is YES, because taxes show up implicitly in the rate of return, i.e. by changing the time-path of tax rates, the government changes the intertemporal incentives of the agents to accumulate savings over time.

Solution to Exercises 2 and 3 provided in class

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Intuition for Some Optimal Taxation Results

Consider a time separable utility function defined over consumption at time $t$ and consumption at time $t + \Delta$. Compute its total differential and set it to zero:

$$du(c_t, c_{t+\Delta}) \simeq u'(c_t)dc_t + u'(c_{t+\Delta})dc_{t+\Delta} = 0 \Rightarrow$$

$$MRS_t = -\frac{\partial c_{t+\Delta}}{\partial c_t} = -\frac{u'(c_t)}{u'(c_{t+\Delta})}$$

The last line defines the marginal rate of substitution $MRS_t$, i.e., the slope of the indifference curve between consumption at time $t$ and consumption at time $t + \Delta$. It is simple to show that as $\Delta \to 0$, we obtain

$$MRS_t = -\frac{u''(c_t)}{u''(c_{t+\Delta})}.$$  

From the FOC’s of the household problem:

$$u'(c_t) = \theta.$$ Therefore,

$$MRS_t = -\frac{\mu(t)}{\mu(t)} = r(t) - \rho.$$  

It is immediate to see that in this model, the marginal rate of substitution $MRS_t$ is simply given by

$$MRT_t = f_k(t) - \delta$$
as saving an extra unit of capital yields $f_k(t) - \delta$ additional units in the next instant. Define now the marginal rate of transformation between zero and time $t$ as

$$MRT_{0,t} = \exp \left[ -\int_0^t (f_k(z) - \delta) dz \right] = \exp \left[ -\int_0^t r(z) dz \right]$$

where the second equality follows from firm’s optimization.

The $MRS_t$ in discounted value terms (at time zero) is simply $r(t)$, hence

$$MRS_{0,t} = \exp \left[ -\int_0^t r(z) dz \right] = MRT_{0,t}$$

which proves that in the standard growth model without distortions, allocations are socially efficient.

Suppose now that we have a capital income tax $\mu > 0$. Then the FOC’s change and in particular

$$\hat{MRS}_{0,t} = \exp(\mu t)$$

so the wedge between $MRT_t$ and $MRS_t$ not only is not zero, as it should be, but it increases exponentially over time. The intuition is that capital is a cumulable factor, so the distortion also cumulates over time, hence even a small tax rate, after a long period of time, has a large effect ("small differences that matter..."). Thus, the capital accumulation margin is a very inefficient margin to distort. On the contrary, the intratemporal margin, linked to the leisure-labor choice

$$\frac{-u_t(t)}{u_t(t)} = w(t) \left[ 1 - r(t) \right],$$

is static and therefore distortions have smaller effects as they do not keep cumulating over time. This provides a powerful intuition on why optimal taxation theory states that only labor (and more in general noncumulable production factors) should carry a positive tax.